This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so. Hand in this exam before you leave.

No problem in this exam requires very long and complicated calculations or proofs. You are probably on the wrong track if you do that.
1. Let matrix $A$ be defined by

$$A = \begin{pmatrix}
7 & -3 & 0 & 0 & 0 \\
0 & 1 & -3 & 0 & 0 \\
0 & 0 & 7 & -3 & 0 \\
0 & 0 & 0 & 1 & -3 \\
-3 & 0 & 0 & 0 & 7
\end{pmatrix}.$$ 

- Calculate $\det(A)$.
- Calculate $\det(A^3)$ without computing $A^3$.

2. Consider the system of linear equations

$$x + 2y + \alpha z = 0,$$
$$-x + z = 0,$$
$$\alpha x - y + z = 0.$$ 

Find the values of $\alpha$ for which the system has a unique solution; infinitely many solutions; and no solutions.

3. Let $B$ be the matrix

$$B = \begin{pmatrix}
1 & 0 & 0 \\
1 & 2 & 0 \\
-1 & 1 & 4
\end{pmatrix}.$$ 

Find a basis for $R^3$ made out of eigenvectors of $B$.

4. Solve the system of differential equations

$$x_1'(t) = 3x_1(t) + 3x_2(t),$$
$$x_2'(t) = -2x_1(t) - 4x_2(t)$$

with initial conditions $x_1(0) = 1$, $x_2(0) = 3$.

Show your work.

5. Let $f(x) = x$ for $0 < x < 1$. Suppose that $f(x)$ can be expanded into an infinite series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x).$$

Find the coefficients $b_n$. 