SOLUTION: ASSIGNMENT 5

4.1.3 Is the subset of $P_2$ given below a subspace? Find a basis if it is.

$$\{ p(t) : p'(1) = p(2) \}$$

($p'$ is the derivative).

**Solution.** Let $S = \{ p(t) : p'(1) = p(2) \}$. For any $p_1, p_2 \in S$, $k_1, k_2 \in \mathbb{R}$, by

$$(k_1 \cdot p_1 + k_2 \cdot p_2)'|_{t=1} = k_1 p_1'(1) + k_2 p_2'(1) = k_1 p_1(2) + k_2 p_2(2) = (k_1 \cdot p_1 + k_2 \cdot p_2)|_{t=2}$$

we checked $k_1 \cdot p_1 + k_2 \cdot p_2 \in S$. Therefore $S$ is a subspace.

For $p(t) = a + bt + ct^2 \in S$,

$$p'(1) = p(2) \Rightarrow b + 2c = a + 2b + 4c \Rightarrow a + b + 2c = 0 \Rightarrow (a, b, c) = (-\lambda - 2\mu, \lambda, \mu)$$

where $\lambda, \mu \in \mathbb{R}$. Then $p(t) = -\lambda - 2\mu + \lambda t + \mu t^2 = \lambda(t - 1) + \mu(t^2 - 2)$, i.e. $S = \text{span}(t - 1, t^2 - 2)$. But $t - 1, t^2 - 2 \in S$ are obviously independent. Therefore $(t - 1, t^2 - 2)$ is a basis of $S$.

4.1.4 Required as in 4.1.3

$$\{ p(t) : \int_0^1 p(t)dt = 0 \}$$

**Solution.** Let $S = \{ p(t) : \int_0^1 p(t)dt = 0 \}$. For any $p_1, p_2 \in S$, $k_1, k_2 \in \mathbb{R}$, by

$$\int_0^1 (k_1 \cdot p_1 + k_2 \cdot p_2)dt = k_1 \int_0^1 p_1(t)dt + k_2 \int_0^1 p_2(t)dt = k_1 0 + k_2 0 = 0$$

we checked $k_1 \cdot p_1 + k_2 \cdot p_2 \in S$. Therefore $S$ is a subspace.

For $p(t) = a + bt + ct^2 \in S$,

$$\int_0^1 p(t)dt = 0 \Rightarrow a + \frac{b}{2} + \frac{c}{3} = 0 \Rightarrow (a, b, c) = (-\lambda - \mu, -\lambda - 2\mu, \lambda, \mu)$$

where $\lambda, \mu \in \mathbb{R}$. Then $p(t) = -\frac{1}{2} - \frac{b}{3} + \lambda t + \mu t^2 = \lambda(t - \frac{1}{2}) + \mu(t^2 - \frac{1}{3})$, i.e. $S = \text{span}(t - \frac{1}{2}, t^2 - \frac{1}{3})$. But $t - \frac{1}{2}, t^2 - \frac{1}{3} \in S$ are obviously independent. Therefore $(t - \frac{1}{2}, t^2 - \frac{1}{3})$ is a basis of $S$.

4.1.21 Find a basis for the space of all diagonal $2 \times 2$ matrices, and determine its dimension.

**Solution.** Any diagonal $2 \times 2$ matrix looks like

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This tells us that $\left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ is a basis because these two matrices are already independent as in $\mathbb{R}^{2 \times 2}$. The dimension is 2.

4.1.22 Find a basis for the space of all diagonal $n \times n$ matrices, and determine its dimension.
Solution. Any diagonal $n \times n$ matrix looks like

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1 E_{11} + \cdots + a_n E_{nn}$$

where $E_{ii}$ is the matrix with entries all 0 except a 1 at the $i$'th diagonal entry. This tells us that $(E_{11}, \ldots, E_{nn})$ is a basis because these $n$ matrices are already independent as in $\mathbb{R}^{n \times n}$. The dimension is $n$.

4.1.24 Find a basis of the space of all upper triangular $3 \times 3$ matrices and determine its dimension.

Solution. Any upper triangular $3 \times 3$ matrix looks like

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{22} & a_{23} & \\ a_{33} & & \end{pmatrix} = a_{11} E_{11} + a_{12} E_{12} + a_{13} E_{13} + a_{22} E_{22} + a_{23} E_{23} + a_{33} E_{33}$$

where $E_{ij}$ denotes the matrix with entries all 0 except for a 1 at the $(i, j)$-th place. This tells us that $(E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33})$ is a basis because these 6 matrices are already independent as in $\mathbb{R}^{3 \times 3}$. The dimension is 6.

4.1.32 Find a basis of the space of all $2 \times 2$ matrices $S$ such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} S = S \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Solution. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the condition becomes

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

which is simplified to be

$$\begin{pmatrix} a + c & b + d \\ a + c & b + d \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 2c & 0 \end{pmatrix}$$

By comparing the entries we get

$$\begin{align*}
     a + c &= 2a \\
     b + d &= 0 \\
     a + c &= 2c \\
     b + d &= 0
\end{align*}$$

so $(a, b, c, d) = (\lambda, \mu, \lambda, -\mu)$. Therefore $S = \begin{pmatrix} \lambda & \mu \\ \lambda & -\mu \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, i.e., $(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mu \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix})$ is a basis of the space of all $S$’s. The dimension is 2.

4.1.50 Find all solutions of the differential equation $f''(x) + 8f'(x) - 20f(x) = 0$.

Solution. Solving the characteristic equation $\lambda^2 + 8\lambda - 20 = 0$ we have two eigenvalues $\lambda_1 = 2, \lambda_2 = -10$. Therefore all solutions are $f(x) = c_1 e^{2x} + c_2 e^{-10x}$, where $c_1, c_2 \in \mathbb{R}$.

Remark. Review how to solve a homogeneous linear ODE in Math 1B. Another way is to follow what we did in Example 18 of Section 4.1.

4.1.58 In this exercise we will show that the function $\cos x$ and $\sin x$ span the solution space $V$ of the differential equation $f''(x) = -f(x)$. 

2
a. Show that if \( g(x) \) is in \( V \), then the function \( g(x)^2 + g'(x)^2 \) is constant.

b. Show that if \( g(x) \) is in \( V \), with \( g(0) = g'(0) = 0 \), then \( g(x) = 0 \) for all \( x \).

c. If \( f(x) \) is in \( V \), then \( g(x) = f(x) - f(0) \cos x - f'(0) \sin x \) is in \( V \) as well. Show then \( f(x) = f(0) \cos x - f'(0) \sin x \).

Solution. a. Taking derivative we have

\[
(g^2 + (g')^2)' = (g^2)' + ((g')^2)' = 2gg' + 2g'g'' = 2g(g + g'')
\]

But \( g \in V \) means \( g'' = -g \), so the right handed side is zero. The derivative being always zero means \( g^2 + (g')^2 \) is constant.

b. Because \( g(x)^2 + g'(x)^2 = C \) is constant, plugging in \( g(0) = g'(0) = 0 \) we know the constant \( C = 0 \). Thus \( g(x)^2 + g'(x)^2 = 0 \), and since the summands are nonnegative, they must both be zero. In particular, \( g(x) = 0 \).

c. \( g(x) \) is in \( V \) because it's a linear combination of \( f, \cos x, \sin x \) which are all in \( V \). But \( g(0) = f(0) - f(0) \cos 0 - f'(0) \sin 0 = 0 \), \( g'(0) = f'(0) + f(0) \sin 0 - f'(0) \cos 0 = 0 \). By b. we know \( g(x) = 0 \), i.e. \( f(x) = f(0) \cos x + f'(0) \sin x \).

4.2.6 Find out if the transformation it is linear, and when linear, if it is isomorphism.

\[
T(M) = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}
\]

from \( \mathbb{R}^{2 \times 2} \) to \( \mathbb{R}^{2 \times 2} \).

Solution. Denote \( \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \) as \( A \). For any \( M_1, M_2 \in \mathbb{R}^{2 \times 2} \) and \( k_1, k_2 \in \mathbb{R} \),

\[
T(k_1M_1 + k_2M_2) = (k_1M_1 + k_2M_2)A = k_1M_1A + k_2M_2A = k_1T(M_1) + k_2T(M_2)
\]

Therefore \( T \) is linear. \( T \) is not an isomorphism because \( A \) is not invertible. In this case \( T \) maps any matrix to a non-invertible matrix so \( \text{im} T \) cannot be \( \mathbb{R}^{2 \times 2} \) (e.g. it avoids all invertible matrices).

4.2.12 Required as in 4.2.6, \( T(c) = c \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \) from \( \mathbb{R} \) to \( \mathbb{R}^{2 \times 2} \).

Solution. Denote \( \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \) as \( A \). For any \( c_1, c_2 \in \mathbb{R} \) and \( k_1, k_2 \in \mathbb{R} \),

\[
T(k_1c_1 + k_2c_2) = (k_1c_1 + k_2c_2)A = k_1c_1A + k_2c_2A = k_1T(c_1) + k_2T(c_2)
\]

Therefore \( T \) is linear. \( T \) cannot an isomorphism because the dimensions of target space and the source space are different.

4.2.14 Required as in 4.2.6, \( T(M) = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} M - \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \) from \( \mathbb{R}^{2 \times 2} \) to \( \mathbb{R}^{2 \times 2} \).

Solution. Denote \( \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \) as \( A \). For any \( M_1, M_2 \in \mathbb{R}^{2 \times 2} \) and \( k_1, k_2 \in \mathbb{R} \),

\[
T(k_1M_1 + k_2M_2) = A(k_1M_1 + k_2M_2) - (k_1M_1 + k_2M_2)A = k_1(AM_1 - M_1A) + k_2(AM_2 - M_2A) = k_1T(M_1) + k_2T(M_2)
\]

Therefore \( T \) is linear. Observe that \( T(I_2) = A I_2 - I_2 A = A - A = 0 \) where \( I_2 \) is the unit matrix, and hence \( T \) is not an isomorphism because \( \ker T \neq \{0\} \) (e.g. it contains \( I_2 \)).

4.2.26 Required as in 4.2.6, \( T(f(t)) = f(-t) \) from \( P_2 \) to \( P_2 \).
Solution. For any $f_1, f_2 \in P_2$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1 f_1 + k_2 f_2)(t) = (k_1 f_1 + k_2 f_2)(-t) = k_1 f_1(-t) + k_2 f_2(-t) = k_1 T(f_1)(t) + k_2 T(f_2)(t)$$

Therefore $T$ is linear. Observe that $T^2$ is identity, i.e. $T(T(f(t))) = T(f(-t)) = f(t)$, so $T$ is the inverse of itself. $T$ is invertible and hence is isomorphism.

4.2.30 Required as in 4.2.6, $T(f(t)) = tf''(t)$ from $P_2$ to $P_2$.

Solution. For any $f_1, f_2 \in P_2$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1 f_1 + k_2 f_2)(t) = (k_1 f_1 + k_2 f_2)(t)' = k_1 f_1'(t) + k_2 f_2'(t) = k_1 T(f_1)(t) + k_2 T(f_2)(t)$$

Therefore $T$ is linear. Observe that $T(c) = t0 = 0$ where $c$ is any constant polynomial, and hence $T$ is not an isomorphism because $\ker T \neq \{0\}$ (e.g. it contains constant polynomials).

4.2.48 Required as in 4.2.6, $T(f(t)) = f'(t)$ from $P$ to $P$.

Solution. For any $f_1, f_2 \in P$ and $k_1, k_2 \in \mathbb{R}$,

$$T(k_1 f_1 + k_2 f_2)(t) = (k_1 f_1 + k_2 f_2)(t)' = k_1 f_1'(t) + k_2 f_2'(t) = k_1 T(f_1)(t) + k_2 T(f_2)(t)$$

Therefore $T$ is linear. Observe that $T(c) = t0 = 0$ where $c$ is any constant polynomial, and hence $T$ is not an isomorphism because $\ker T \neq \{0\}$ (e.g. it contains constant polynomials).

Remark. But in this case, $\text{im} T = P$, i.e. it is surjective. The explanation is that $P$ is infinite dimensional.

4.2.56 Find image, rank, kernel and nullity of the transformation in 4.2.30

Solution. Let $f(t) = a + bt + ct^2$, then $T(f(t)) = t(a + bt + ct^2)' = bt + 2ct^2$. Thus $\text{im} T = \text{span}\{t, 2t^2\}$. Consider $bt + 2ct^2 = 0$ as a polynomial, i.e. all coefficients are zero, so $(a, b, c) = (\lambda, 0, 0)$ where $\lambda \in \mathbb{R}$. Thus $\ker T = \text{span}\{1\}$. The rank is $\dim \text{im} T = 2$, and the nullity is $\dim \ker T = 1$.

4.2.67 For which constants $k$ is the linear transformation

$$T(M) = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} M - M \begin{pmatrix} 3 & 0 \\ 0 & k \end{pmatrix}$$

an isomorphism from $\mathbb{R}^{2 \times 2}$ to $\mathbb{R}^{2 \times 2}$?

Solution. The most standard way to understand a linear transformation is to find out its matrix under a favorable basis. For simplicity we write the standard basis of $\mathbb{R}^{2 \times 2}$, namely $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ as $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$. Then

$$T(\vec{e}_1) = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = -\vec{e}_1$$

Similarly, $T(\vec{e}_2) = (2 - k)\vec{e}_2, T(\vec{e}_3) = 3\vec{e}_1 + \vec{e}_3, T(\vec{e}_4) = 3\vec{e}_2 + (4 - k)\vec{e}_4$. Therefore,
\[ T \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = T(a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3 + d\hat{e}_4) \]
\[ = T \left( \begin{array}{cccc} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & \hat{e}_4 \end{array} \right) \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \]
\[ = \left( \begin{array}{cccc} T(\hat{e}_1) & T(\hat{e}_2) & T(\hat{e}_3) & T(\hat{e}_4) \end{array} \right) \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \]
\[ = \left( \begin{array}{cccc} -\hat{e}_1 & (2-k)\hat{e}_2 & 3\hat{e}_1 + e_3 & 3\hat{e}_2 + (4-k)\hat{e}_4 \end{array} \right) \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \]
\[ = \left( \begin{array}{cccc} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & \hat{e}_4 \end{array} \right) \left( \begin{array}{cccc} -1 & 0 & 3 & 0 \\ 0 & 2-k & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4-k \end{array} \right) \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \]

The matrix here is the matrix of \( T \) under the standard basis. Clearly it is an isomorphism if and only if \( k \) is neither 2 nor 4.

**Remark.** The third equality of the last computation formally looks like the associativity, which turns out to be true for linear transformation. Anyway, this way of computation helps us a lot to get a correct matrix of a transformation under a given basis. In fact, we may just formally compute \( T(\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4) = (T(\hat{e}_1) T(\hat{e}_2) T(\hat{e}_3) T(\hat{e}_4)) A \) to get the matrix \( A \) of \( T \) under any basis \( (\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4) \) without writing down the coordinate column vector (cf. the solution of 4.2.68 as an example).

**4.2.68** *For which constants \( k \) is the linear transformation \( T(M) = M \) an isomorphism from \( \mathbb{R}^{2 \times 2} \) to \( \mathbb{R}^{2 \times 2} \)?*

**Solution.** Use the same method as in 4.2.67, we will find \( T(\hat{e}_1) = 3\hat{e}_1, T(\hat{e}_2) = -\hat{e}_2, T(\hat{e}_3) = (5-k)\hat{e}_3, T(\hat{e}_4) = (1-k)\hat{e}_4 \). Therefore,

\[ T \left( \begin{array}{cccc} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & \hat{e}_4 \end{array} \right) = \left( \begin{array}{cccc} T(\hat{e}_1) & T(\hat{e}_2) & T(\hat{e}_3) & T(\hat{e}_4) \end{array} \right) \]
\[ = \left( \begin{array}{cccc} 3\hat{e}_1 & -\hat{e}_2 & (5-k)\hat{e}_3 & (1-k)\hat{e}_4 \end{array} \right) \]
\[ = \left( \begin{array}{cccc} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 & \hat{e}_4 \end{array} \right) \left( \begin{array}{cccc} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5-k & 0 \\ 0 & 0 & 0 & 1-k \end{array} \right) \]

The matrix of \( T \) is invertible if and only if \( k \) is neither 5 nor 1.