Solutions: Assignment 10

7.2.2 The eigenvalues are 2 and 1, each with multiplicity 2.

7.2.8 The polynomial is $(\lambda + 3)\lambda^2$, so the eigenvalues are 0 with multiplicity 2 and $-3$ with multiplicity 1.

7.2.14

$$\det\left(\lambda I - \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda I - B & -C \\ 0 & \lambda I - D \end{bmatrix}\right) = \det(\lambda I - B)\det(\lambda I - D)$$

So the eigenvalues of $A$ is equivalent to being an eigenvalue of $B$ or $D$.

7.2.16 Characteristic polynomial is $\lambda^2 - (a + c)\lambda + (a - c)^2 + 4b^2 > 0$. So there are two real roots.

7.2.20

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\text{lambda}_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

$$\det\left(\lambda I - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \lambda^2 - (a + d)\lambda + (ad - bc)$$

So $\lambda_1 + \lambda_2 = a + d = \text{tr}(A)$.

7.2.24 eigenvalues are $\frac{1}{4}$ and 1.

7.3.6 eigenvalues are $\frac{7}{2} \pm \frac{\sqrt{57}}{2}$. eigen basis is $\left[\frac{3}{2} \pm \frac{\sqrt{57}}{2}\right]$.

7.3.20 The eigenspace for 1 is the kernel of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which row reduces to $\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So the eigenspace has dimension 1 unless $a = 0$ and then it has dimension 2. The eigenspace for 2 is the kernel of $\begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix}$ which is always 1 dimensional.

7.3.24

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

7.3.34 (a) From $SB = AS$ and $B\overline{f} = \overline{0}$, we get $A(S\overline{f}) = SB\overline{f} = \overline{0}$.

(b) Same as (a) with $B$ and $A$ switched and $S$ and $S^{-1}$ switched.

(c) Since $null(A) = null(B)$ and $A$ and $B$ are the same size, $rk(A) = rk(B)$.1
7.4.4 The eigenvalues are 7 with eigenvector \[
\begin{bmatrix}
1 \\
3
\end{bmatrix}
\] and 0 with eigenvector \[
\begin{bmatrix}
2 \\
-1
\end{bmatrix}
\]. So \(A = SDS^{-1}\) where \(D = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\) and \(S = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\).

7.4.16 The eigenvalues are 1 with eigenvector \[
\begin{bmatrix}
0 \\
-1
\end{bmatrix}
\] The eigenvalues are 2 with eigenvector \[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\] and 3 with eigenvector \[
\begin{bmatrix}
2 \\
0 \\
1
\end{bmatrix}
\]. So \(A = SDS^{-1}\) where \(D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\) and \(S = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\).

7.4.22 If \(b = 1\) and \(a \neq 0\), then there is only one eigenspace and it is of dimension 1, so it is not diagonalizable. In all other cases it is diagonalizable.

7.2.40

\[T(y) = 5y' - 3y = \lambda y\]
\[y' - \frac{3 - \lambda}{5} y = 0\]
\[y = ce^{(3-\lambda)t/5}\]

So every real number is an eigenvalue with an eigenspace of dimension 1.

7.4.46

\[T(x_0, x_1, \ldots) = (x_0, x_2, \ldots) = \lambda (x_0, x_1, \ldots)\]

The odd terms can be anything. For \(i > 0\), \(x_{2i} = \lambda x_i\). If \(\lambda = 1\) then \(x_0\) can be anything. Otherwise \(x_0 = 0\).

7.4.56 Since \(M = SNS^{-1}\),

\[
\lambda^n \det(\lambda I - AB) = \det(\lambda I - M) = \det(\lambda I - SNS^{-1}) = \det(S(\lambda I - N)S^{-1}) = \det(S)\det(\lambda I - N)\det(S)^{-1} = \det(\lambda I - N) = \lambda^n \det(\lambda I - BA)
\]

So

\[\det(\lambda I - AB) = \det(\lambda I - AB)\]

7.4.62

\[T(y) = y'' + ay' + by = \lambda y\]
\[y'' + ay' + (b - \lambda)y = 0\]

This differential equation always has a solution space of dimension exactly 2. So every number is an eigenvalue with a 2-dimensional eigenspace.