Math 54 - Homework 8 Solutions

5.4.2

\[ \ker(A^T) = \ker \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \right) \]

\[ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0 \]

Basis for \( \ker(A^T) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \)

For the sketch illustrating \( (\text{Im}(A))^\perp = \ker(A^T) \), \( \text{Im}(A) \) is the plane spanned by \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). Notice that \( \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \) is orthogonal to \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)

5.4.7

A is a \( n \times n \) symmetric matrix. This implies that \( A = A^T \)

\( (\text{Im}(A))^\perp = \ker(A^T) = \ker(A) \)

Or alternatively, \( \text{Im}(A) = (\ker(A))^\perp \)
\( A \) has dimension \( m \times n \)

\( y \) has dimension \( m \times 1 \)

\( \ker(A) = 0 \Rightarrow A^T A \) is invertible

Least squares solution of \( L(x) = y \) is \( (A^T A)^{-1} A^T y \)

(a)

\[
L^+(y) = (A^T A)^{-1} A^T y
\]

\[
A^+ = (A^T A)^{-1} A^T
\]

It is easy to verify that \( L^+ \) is linear

\[
L^+(y + z) = L^+(y) + L^+(z)
\]

\[
L^+(k y) = k L^+(y)
\]

(b)

If \( A \) is invertible, then \( (A^T A)^{-1} = A^{-1} (A^T)^{-1} \)

then, \( A^+ = (A^T A)^{-1} A^T \)

\[
= A^{-1} (A^T)^{-1} A^T
\]

\[
= A^{-1}
\]

If \( L \) is invertible, \( L^+ = L^{-1} \)

(c)

\[
L^+(L(x))
\]

\[
L^+(Ax)
\]

\[
(A^T A)^{-1} A^T Ax
\]

\[
= x
\]

(d)

\[
L(L^+(y))
\]

\[
= L((A^T A)^{-1} A^T y)
\]

\[
= A(A^T A)^{-1} A^T y
\]
5.4.10 (a)  

Let \( x \) be a solution to the system \( x = x_h + x_0 \) where \( x_h \in \text{ker}(A) \), \( x_0 \in (\text{ker}(A))^\perp \)

\[
\begin{align*}
Ax &= b \\
A(x_h + x_0) &= b \\
Ax_h + Ax_0 &= b \\
Ax_0 &= b
\end{align*}
\]

(b)  

\[
\begin{align*}
Ax_0 &= b \\
Ax_1 &= b
\end{align*}
\]

where \( x_0, x_1 \in (\text{ker}(A))^\perp \)

\[
\begin{align*}
A(x_0 - x_1) &= Ax_0 - Ax_1 \\
&= b - b = 0
\end{align*}
\]

which is a contradiction. It would imply that \( x_0 - x_1 \in \text{ker}(A) \)

(c)  

If \( x_1 \) is a solution, it can be written as \( x_1 = x_{\text{ker}(A)} + x_{(\text{ker}(A))^\perp} \)

However, in (b) we established that \( x_{(\text{ker}(A))^\perp} \) has only one possible value, \( x_0 \)

Thus, \( x_1 = x_{\text{ker}(A)} + x_0 \)

Look at \( \|x_1\|^2 = (x_{\text{ker}(A)} + x_0) \cdot (x_{\text{ker}(A)} + x_0) \)

\[
= x_0 \cdot x_0 + \text{something positive} = \|x_0\|^2 + \text{something positive}
\]

It follows that \( \|x_1\|^2 > \|x_0\|^2 \)
\[ A \text{ is an } m \times n \text{ matrix} \]
\[ ker(A) = \{0\} \]
\[ A^T A \text{ is invertible by fact 5.4.2b} \]
\[ B \cdot A = I \]
\[ A^T \cdot B^T = I \]
\[ A^T A (A^T A)^{-1} = I \]

Let \( B = (A(A^T A)^{-1})^T = (A^T A)^{-1} A^T \)
5.5.2

\[ <f, g + h> = <g + h, f> = <g, f> + <h, f> = <f, g> + <f, h> \]

5.5.4 (a)

\[ <A, B> = tr(A^T B) \]

\[ = tr \left( \begin{bmatrix} a_1 & \cdots & a_n \\ \vdots \\ b_1 \\ \vdots \\ b_n \end{bmatrix} \right) \]

\[ = tr([a_1 \cdot b_1 + \ldots + a_n \cdot b_n]) \]

You can think of this as a one by one matrix.

The trace of a square matrix is the sum of its diagonal matrix.

Thus, \( <A, B> \) is the dot product of \( A \) and \( B \)

(b)

\[ <A, B> = tr \left( \begin{bmatrix} a_1 \\ \vdots \\ a_m \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \right) \]

\[ = a_1 \cdot b_1 + \ldots a_m \cdot b_m \]

5.5.10

\[ <f, g> = \frac{1}{2} \int_{-1}^{1} f(t)g(t) \, dt \]

\[ <f, g> = \frac{1}{2} \int_{-1}^{1} t(at^2 + bt + c) \, dt \]

\[ = \frac{1}{2} \left[ \frac{9t^4}{4} + \frac{bt^3}{3} + \frac{ct^2}{2} \right]_{-1}^{1} \]

set \( <t, g(t)> = 0 \), we get \( \frac{2b}{3} = 0 \) or \( b = 0 \)

\[ g(t) = at^2 + c \]

A basis for the space of all functions in \( P_2 \) that are orthogonal to \( f(t) = t \) is \{1, t^2\}.

Apply Gram-Schmidt to the basis \{v_1, v_2\}
\[ u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\int_{-1}^{1} dt}} = 1 \]

\[ u_2 = \frac{v_2}{\|v_2\|} \quad \text{where} \quad v_2 = v_2 - (u_1 \cdot v_2)u_1 = t^2 - \frac{1}{3} \]

\[ \|v_2\|^2 = <t^2 - \frac{1}{3}, t^2 - \frac{1}{3}> \]

\[ = <t^2, t^2> + 2 <t^2, -\frac{1}{3}> + <\frac{1}{3}, -\frac{1}{3}> \]

\[ = \frac{1}{5} - \frac{2}{9} + \frac{1}{9} = \frac{1}{5} - \frac{1}{9} = \frac{9 - 5}{45} = \frac{4}{45} \]

\[ u_2 = \frac{45}{4}(t^2 - \frac{1}{3}) \]

5.5.12

\[ f(t) = |t| \]

\[ b_k = <f(t), \sin(kt)> \]

\[ = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \sin(kt) \, dt \]

\[ c_k = <f(t), \cos(kt)> = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \cos(kt) \, dt \]

\[ a_0 = <f(t), \frac{1}{\sqrt{2}} > = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |t| \, dt = \frac{1}{\sqrt{2\pi}} \pi^2 \]

5.5.16

\[ P_1 \text{ with inner product } <f, g> = \int_{0}^{1} f(t) g(t) \, dt \]

(a)

Basis for \( P_1 = \{v_1, v_2\} = \{1, t\} \)

\[ u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\int_{0}^{1} dt}} = 1 \]

\[ u_2 = \frac{v_2}{\|v_2\|} = \sqrt{12}(t - \frac{1}{2}) \]

where \( v_2^\perp = v_2 - (u_1 \cdot v_2)u_1 = (t - \frac{1}{2}) \)
\[ \|v_2^1\| = \sqrt{\int_0^1 t^2 - t + \frac{1}{4} \, dt} \]
\[ = \sqrt{\left[ \frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{4} t \right]_0^1} \]
\[ = \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} \]
\[ = \sqrt{\frac{1}{12}} \]

(b)

We need to find \( proj_{p_1}(f(t)) \)

\[ proj_{p_1}(f) = < u_1, f > u_1 + < u_2, f > u_2 \]

\[ = < 1, t^2 > 1 + \left( \sqrt{\frac{12}{2}} \left( t - \frac{1}{2} \right), t^2 \right) \sqrt{\frac{12}{2}}(t - \frac{1}{2}) \]

5.2
This is **FALSE**. \((AB)^T = B^TA^T \neq A^TB^T\) unless \(B^T\) and \(A^T\) commute.

5.4
This is **TRUE**.

\(A\) and \(S\) are orthogonal

\[ \|S^{-1}ASx\| = \|SS^{-1}ASx\| \]

\[ = \|ASx\| \]

\[ = \|Sx\| \]

\[ = \|x\| \]

5.10
This is **TRUE**.

\[ AA^{-1} = I \]

\[ (AA^{-1})^T = I^T \]

\[ (A^{-1})^TA^T = I \]

5.20
\(ker(A) = 0\) only guarantees that the column vectors are linearly independent. However, by fact 5.3.10, we also necessitate that the column vectors have length 1 and are
orthogonal to each other.

5.22 This is **TRUE** because \( \mathbb{R}^n \) is an inner product space, and also by the Gram-Schmidt process.

\[
6.1.2 \\
\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 2*5 - 3*4 = -2
\]

\[
6.1.6 \\
\det \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} = 6*4*1 = 24
\]

\[
6.1.12 \\
\det \begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix} = 4 - k^2
\]

This is invertible only if \( k \neq \pm 2 \)

\[
6.1.22 \\
\det \begin{bmatrix} \cos(k) & 1 & -\sin(k) \\ 0 & 2 & 0 \\ \sin(k) & 0 & \cos(k) \end{bmatrix}
\]

\[
= \cos(k)(2\cos(k)) - 1*0 - \sin(k)(-2\sin(k))
\]

\[
= 2(\cos^2(k) + \sin^2(k)) = 2
\]

\[
6.1.40 \\
There are lots of zeroes everywhere. This is good.
\]

\[
\det = 3* -2* -4* -5
\]

\[
= -120
\]

\[
6.1.44 \\
For n by n matrix A: \det(kA) = k^n * \det(A).
\]

Proof by induction works here. When \( n=1 \) or \( n=2 \), this is easy to see. Assume true for \( n=m-1 \), show statement is true for \( n=m \).

\[
6.1.55 \text{ (a)}
\]

Notice that \( d_4 = d_3 - \det \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \)

\[
= d_3 - d_2
\]
In general, \( d_n = d_{n-1} - d_{n-2} \)

(b) \[ d_1 = 1; d_2 = 0 \]

The rest follows from the recursive formula.

\[ 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, \ldots \]

Note that the sequence repeats

It follows from part (a) that \( 100 \mod 6 = 4 \) so, \( d_{100} = d_4 = -1 \)