5.1.16 Consider the vectors

\[ u_1 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \quad u_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \]

in \( \mathbb{R}^4 \). Can you find a vector \( u_4 \) such that \( u_1, u_2, u_3, u_4 \) are orthonormal? If so, how many such vectors are there?

Note that \( u_1, u_2, u_3 \) are already orthonormal, so we just need to find \( u_4 \), which must satisfy \( u_1 \cdot u_4 = u_2 \cdot u_4 = u_3 \cdot u_4 = 0 \) and \( u_4 \cdot u_4 = 1 \). If \( u_4 = (a, b, c, d)^T \), then the first three conditions give the linear equations

\[
\begin{align*}
\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d &= 0 \\
\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c - \frac{1}{2}d &= 0 \\
\frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d &= 0
\end{align*}
\]

Row reducing these equations leads to the system

\[
\begin{align*}
a + b + c + d &= 0 \\
b + d &= 0 \\
c + d &= 0
\end{align*}
\]

which has general solution \( u_4 = (a, b, c, d)^T = (t, -t, -t, t)^T \). Any such vector has dot product zero with \( u_1, u_2, u_3 \). For an orthonormal basis, we want \( u_4 \cdot u_4 = 1 \), which gives \( t = \pm \frac{1}{2} \), so there are 2 choices for \( u_4 \), namely

\[
\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}
\]

5.1.20 (See the book for the full statement of the problem, which involves finding a least squares estimate for a line \( y = mx \)).

Let \( \mathbf{x} \) be the vector of x coordinates and \( \mathbf{y} \) be the vector of y coordinates. Geometrically, \( \{m\mathbf{x}\} \) is a 1 dimension vector space. We which to minimize the distance from \( \mathbf{y} \) into this space; this is accomplished when \( m\mathbf{x} \) is the projection of \( \mathbf{y} \) into this space. Note that \( \frac{\mathbf{x}}{||\mathbf{x}||} \) is a unit vector in this space, so the corresponding projection should be \( (\mathbf{y} \cdot \frac{\mathbf{x}}{||\mathbf{x}||}) \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{\mathbf{y} \cdot \mathbf{x}}{||\mathbf{x}||^2} \). In particular, we should take \( m = \frac{\mathbf{y} \cdot \mathbf{x}}{||\mathbf{x}||^2} = \frac{4182.9}{198.53^2} = 0.106 \ldots \).
If you draw the relevant line in the picture, most points will be near the line (even if none are on the line).

5.1.22 Consider a basis $v_1, v_2, \ldots, v_m$ of a subspace $V$ of $\mathbb{R}^n$. Show that a vector $x$ in $\mathbb{R}^n$ is orthogonal to $v$ if and only if it is orthogonal to all the vectors $v_1, \ldots, v_m$.

If $x$ is orthogonal to all the vectors $v_1, \ldots, v_m$, then for any $v \in V$, we can write $v = c_1 v_1 + \cdots + c_m v_m$. Then $x \cdot v = x \cdot (c_1 v_1 + \cdots + c_m v_m) = c_1 (x \cdot v_1) + \cdots + c_m (x \cdot v_m) = 0$ (since each $x \cdot v_i = 0$). So $x$ is orthogonal to every vector in $V$.

If $x \cdot v = 0$ for all $v \in V$, then this applies in particular to the vectors $v_1, \ldots, v_m \in V$.

5.1.24 Complete the proof of Fact 5.1.4: Orthogonal Projection are linear transformations.

We must show that $\text{proj}_V(x + y) = \text{proj}_V(x) + \text{proj}_V(y)$ and $\text{proj}_V(cx) = c \text{proj}_V(x)$.

From page 189, if $u_1, \ldots, u_m$ are an orthonormal basis for $V$, then $\text{proj}_V(x + y) = (u_1 \cdot (x + y)) u_1 + \cdots + (u_m \cdot (x + y)) u_m = ((u_1 \cdot x) + (u_1 \cdot y)) u_1 + \cdots + ((u_m \cdot x) + (u_m \cdot y)) u_m = [((u_1 \cdot x) u_1 + \cdots + (u_m \cdot x) u_m] + [((u_1 \cdot y) u_1 + \cdots + (u_m \cdot y) u_m] = \text{proj}_V(x) + \text{proj}_V(y)$. Similarly $\text{proj}_V(cx) = (u_1 \cdot cx) u_1 + \cdots + (u_m \cdot cx) u_m = c[(u_1 \cdot x) u_1 + \cdots + (u_m \cdot x) u_m] = c \text{proj}_V(x)$.

5.1.30 Consider a subspace $V$ or $\mathbb{R}^n$ and a vector $x$ in $\mathbb{R}^n$. Let $y = \text{proj}_V(x)$. What is the relation ship between $||y||^2$ and $y \cdot x$?

The two quantities are equal. One can see this by writing $x = y + w$; then $y$ is the parallel component to $V$ and $w$ is orthogonal to $V$, and in particular $y \cdot w = 0$. Then $x \cdot y = (y + w) \cdot y = y \cdot y = ||y||^2$ as desired.

5.2.2 Perform Gram-Schmidt on the vectors

$$
\begin{pmatrix}
6 \\
3 \\
2
\end{pmatrix}
\quad
\begin{pmatrix}
2 \\
-6 \\
3
\end{pmatrix}
$$

We have

$$
r = \frac{m||x||}{||y||} \text{ from our formula. (Note that } ||y||/||x|| \text{ is the expected slope if all of the points did lie on a line, so } r \text{ can be thought of as a ratio of two slopes.)}
$$

If you draw the relevant line in the picture, most points will be near the line (even if none are on the line).
\[ u_1 = \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix} \]

\[ v_2^\perp = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} - 0 \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \]

\[ u_2 = \begin{pmatrix} 2/7 \\ -6/7 \\ 3/7 \end{pmatrix} \]

5.2.14 Perform Gram-Schmidt on the vectors

\[
\begin{pmatrix}
1 & 0 & 1 \\
7 & 7 & 8 \\
1 & 2 & 1 \\
7 & 7 & 6
\end{pmatrix}
\]

We have

\[ u_1 = \begin{pmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{pmatrix} \]

\[ v_2^\perp = \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix} - 10 \begin{pmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]

\[ u_2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \]

\[ v_3^\perp = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix} - 10 \begin{pmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{pmatrix} - 0 \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \]

\[ u_3 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \]

5.2.16 Find the QR decomposition of the matrix

\[
\begin{pmatrix}
6 & 2 \\
3 & -6 \\
2 & 3
\end{pmatrix}
\]
From the work in problem 2, we can take $Q$ to be

$$
\begin{pmatrix}
\frac{6}{7} & \frac{2}{7} \\
\frac{3}{7} & -\frac{6}{7} \\
\frac{2}{7} & \frac{3}{7}
\end{pmatrix}
$$

$R$ is calculated by computing various dot products of the column vectors of $Q$ with the column vectors of the original matrix, so we get $R$ to be

$$
\begin{pmatrix}
7 & 0 \\
0 & 7
\end{pmatrix}
$$

5.2.28 Find the $QR$ decomposition of the matrix

$$
\begin{pmatrix}
1 & 0 & 1 \\
7 & 7 & 8 \\
1 & 2 & 1 \\
7 & 7 & 6
\end{pmatrix}
$$

From the work in problem 2, we can take $Q$ to be

$$
\begin{pmatrix}
\frac{1}{10} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{7}{10} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{10} & \frac{1}{\sqrt{2}} & 0 \\
\frac{7}{10} & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix}
$$

$R$ is calculated by computing various dot products of the column vectors of $Q$ with the column vectors of the original matrix, so we get $R$ to be

$$
\begin{pmatrix}
10 & 10 & 10 \\
0 & \sqrt{2} & 0 \\
0 & 0 & \sqrt{2}
\end{pmatrix}
$$

5.2.32 Find an orthonormal basis of the plane $x_1 + x_2 + x_3 = 0$.

First we find a basis for the plane by backsolving the equation. For example, one such basis is

$$
v_1 =\begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix} \quad v_2 =\begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix}
$$

Next we apply Gram-Schmidt to this basis to make it orthonormal.
\[ u_1 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \]

\[ v_2^\perp = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \]

\[ u_2 = \begin{pmatrix} -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \\ -1/\sqrt{6} \end{pmatrix} \]

\[ u_1, u_2 \text{ form an orthonormal basis for the plane } x_1 + x_2 + x_3 = 0. \]

5.3.2 Is \[
\begin{pmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{pmatrix}
\] orthogonal?

Yes. Either one can note that the columns are orthogonal vectors, or one can compute \( A^T A \) and see that you get the identity matrix.

5.3.10 If \( A \) and \( B \) are orthogonal matrices, is \( B^{-1} AB \) orthogonal also?

Yes. By fact 5.3.4 a, \( B^{-1} \) is also orthogonal, and then applying Fact 5.3.4 b several times shows that the product \( B^{-1} AB \) of three orthogonal matrices is an orthogonal matrix. (Alternatively, one can show \( (B^{-1} AB)^T = (B^{-1} AB)^{-1} \) directly.)

5.3.20 If \( A \) and \( B \) are symmetric, is \( AB^2 A \) symmetric?

Yes. \( (AB^2 A)^T = A^T (B^2)^T A^T = AB^2 A \) as required.

5.3.22 If \( B \) is an \( n \times n \) matrix, is \( BB^T \) symmetric?

Yes. \( (BB^T)^T = (B^T)^T B^T = BB^T \) as desired.

5.3.40 Consider the subspace \( W \) of \( \mathbb{R}^4 \) spanned by the vectors

\[ v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix} \]

Find the matrix of orthogonal projection onto \( W \).
We start by finding an orthonormal basis of $W$ using Gram-Schmidt. We get

\[
\begin{align*}
\mathbf{u}_1 &= \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \\
\mathbf{v}_2^\perp &= \begin{pmatrix} 1 \\ 9 \\ -5 \\ 3 \end{pmatrix} - 4 \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -7 \end{pmatrix} \\
\mathbf{u}_2 &= \begin{pmatrix} -1/10 \\ 7/10 \\ -7/10 \\ 1/10 \end{pmatrix}
\end{align*}
\]

So setting

\[
\begin{pmatrix} \frac{1}{2} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{7}{10} & -\frac{1}{2} \\ \frac{7}{10} & \frac{1}{2} & \frac{1}{10} & -\frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} & \frac{1}{10} & \frac{7}{10} \end{pmatrix}
\]

We have that the matrix of orthogonal projection onto $W$ is $QQ^T$ or

\[
\begin{pmatrix}
26 & 18 & 32 & 24 \\
18 & 18 & 40 & 24 \\
32 & 40 & 74 & 32 \\
24 & 24 & 32 & 24 \\
26 & 18 & 32 & 24
\end{pmatrix}
\]

5.3.54 Find the dimension of the space of all skew symmetric matrices.

The dimension is $\frac{n^2-n}{2}$, since such a matrix must have all 0’s on the diagonal, and then is determined by the values in the strictly upper triangular part of the matrix.