Solutions HW 13

9.4.2 Write the given system in matrix form \( x' = Ax + f \)

\[
\begin{align*}
  r'(t) &= 2r(t) + \sin t \\
  \theta'(t) &= r(t) - \theta(t) + 1
\end{align*}
\]

We write this as

\[
\begin{pmatrix}
  r'(t) \\
  \theta'(t)
\end{pmatrix} =
\begin{pmatrix}
  2 & 0 \\
  1 & -1
\end{pmatrix}
\begin{pmatrix}
  r(t) \\
  \theta(t)
\end{pmatrix} +
\begin{pmatrix}
  \sin(t) \\
  1
\end{pmatrix}
\]

9.4.4 Write the given system in matrix form \( x' = Ax + f \)

\[
\begin{align*}
  \frac{dx}{dt} &= x + y + z \\
  \frac{dy}{dt} &= 2x - y + 3z \\
  \frac{dz}{dt} &= x + 5z
\end{align*}
\]

We write this as

\[
\begin{pmatrix}
  \frac{dx}{dt} \\
  \frac{dy}{dt} \\
  \frac{dz}{dt}
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & 1 \\
  2 & -1 & 3 \\
  1 & 0 & 5
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]

9.4.8 Rewrite \( \frac{d^3 y}{dt^3} - \frac{dy}{dt} + y = \cos(t) \) as a first order system in normal form.

Note that the equation says that \( \frac{d^3 y}{dt^3} = \frac{dy}{dt} - y + \cos(t) \). Setting \( x_1 = y, x_2 = \frac{dy}{dt}, x_3 = \frac{d^2 y}{dt^2} \), (so \( \frac{d^3 y}{dt^3} = x_2 - x_1 + \cos(t) \)) we get

\[
\begin{pmatrix}
  x_1' \\
  x_2' \\
  x_3'
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} +
\begin{pmatrix}
  0 \\
  0 \\
  \cos(t)
\end{pmatrix}
\]
9.4.10 Write the given system as a set of scalar equations

\[
x' = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} x + e^t \begin{pmatrix} t \\ 1 \end{pmatrix}
\]

This becomes the equations

\[
x'_1 = 2x_1 + x_2 + te^t \\
x'_2 = -x_1 + 3x_2 + e^t
\]

9.4.16 Determine whether the given vector functions are linearly dependent or independent on the interval \((-\infty, \infty)\)

\[
\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}
\]

We compute the Wronskian

\[
\det \begin{pmatrix} \sin t & \sin 2t \\ \cos t & \cos 2t \end{pmatrix} = \sin t \cos 2t - \sin 2t \cos t = -\sin t
\]

where the last step can be deduced by using trig identities. Since \(-\sin t\) is not identically 0, the vector functions are linearly independent. (Alternatively, one can check that the Wronksian is nonzero at a point such as \(t = \frac{\pi}{2}\).)

9.4.18 Determine whether the given vector functions are linearly dependent or independent on the interval \((-\infty, \infty)\)

\[
\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}, \begin{pmatrix} t^2 \\ 0 \\ t^2 \end{pmatrix}
\]

These functions are linearly independent, since a linear relations requires finding nonzero constants \(c_1, c_2, c_3\) such that \(c_1 + c_2 t + c_3 t^2 = 0\). But \(1, t, t^2\) are linearly independent, so no such constants exist.

Note that even though the vector functions are linearly independent, their Wronksian is still zero.

9.4.22 Determine whether the given functions form a fundamental solution set to an equation \(x'(t) = Ax\). If they do, find a fundamental matrix for the system and give a general solution.
\[
x_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad x_2 = \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \end{pmatrix}, \quad x_3 = \begin{pmatrix} -\cos t \\ \sin t \\ \cos t \end{pmatrix}
\]

We start by computing the Wronksian

\[
\det \begin{pmatrix}
  e^t & \sin t & -\cos t \\
  e^t & \cos t & \sin t \\
  e^t & -\sin t & \cos t
\end{pmatrix} = e^t (\cos^2 t + \sin^2 t) - e^t (\sin t \cos t - \sin t \cos t) + e^t (\sin^2 t + \cos^2 t) = 2e^t
\]

Since this is nowhere 0, the solutions are linearly independent and form a fundamental set. A fundamental matrix is

\[
\begin{pmatrix}
  e^t & \sin t & -\cos t \\
  e^t & \cos t & \sin t \\
  e^t & -\sin t & \cos t
\end{pmatrix}
\]

and a general solution is \(c_1 x_1 + c_2 x_2 + c_3 x_3\).

9.4.24 Verify that the vector functions

\[
x_1 = \begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix}, \quad x_2 = \begin{pmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{pmatrix}
\]

are solutions to the homogenous system

\[
x' = Ax = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} x,
\]
on \((−∞, ∞)\) and that

\[
x_p = \begin{pmatrix} 5t + 1 \\ 2t \\ 4t + 2 \end{pmatrix}
\]
is a particular solution to

\[
x' = Ax + \begin{pmatrix} -9t \\ 0 \\ -18t \end{pmatrix} = A x + f(t)
\]
Find a general solution to \( x' = Ax + f(t) \).

We check directly that

\[
\begin{align*}
x'_1 &= \begin{pmatrix} 3e^{3t} \\ 0 \\ 3e^{3t} \end{pmatrix} = Ax_1 \\
x'_2 &= \begin{pmatrix} -3e^{3t} \\ 3e^{3t} \\ 0 \end{pmatrix} = Ax_2 \\
x'_3 &= \begin{pmatrix} 3e^{-3t} \\ -3e^{-3t} \\ 0 \end{pmatrix} = Ax_3 \\
x'_p &= \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} = Ax_p + f(t)
\end{align*}
\]

A general solution to \( x' = Ax + f(t) \) is \( c_1x_1 + c_2x_2 + c_3x_3 + x_p \).

9.4.25 Prove that the operator \( L[x] = x' - Ax \) is a linear operator.

We must show \( L[x+y] = L[x] + L[y] \) and \( L[cx] = cL[x] \).

\[
\begin{align*}
L[x+y] &= (x+y)' - A(x+y) = x' + y' - Ax - Ay = (x' - Ax) + (y' - Ay) = L[x] + L[y] \\
L[cx] &= (cx)' - A(cx) = cx' - cAx = c(x' - Ax) = cL[x]
\end{align*}
\]

9.4.26 Let \( X(t) \) be a fundamental matrix for the system \( x' = Ax \). Show that \( x(t) = X(t)X^{-1}(t_0)x_0 \) is the solution to the initial value problem \( x' = Ax, x(t_0) = x_0 \).

Since \( x(t) \) is a linear combination of the columns of the fundamental matrix, we just need to check that it satisfies the initial conditions. But \( x(t_0) = X(t_0)X^{-1}(t_0)x_0 = Ix_0 = x_0 \) as desired, so \( x(t) \) is the desired solution.

9.5.6 Find eigenvalues and eigenvectors of the matrix

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

We start by computing the characteristic polynomial.
det \( \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \) = -\( \lambda^3 + 3\lambda + 2 = (2 - \lambda)(1 + \lambda)^2 \)

So the eigenvalues are 2 and -1.

For \( \lambda = -1 \) we must find the kernel of

\[
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

Row reducing we get

\[
\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

which gives eigenvectors

\[
\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}
\]

For \( \lambda = 2 \) we must find the kernel of

\[
\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}
\]

Row reducing we get

\[
\begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}
\]

which gives the eigenvector

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

9.5.10 Find all eigenvalues and eigenvectors of
We start by computing the characteristic polynomial.

\[
\begin{vmatrix}
1 - \lambda & 2 & -1 \\
0 & 1 - \lambda & 1 \\
0 & -1 & 1 - \lambda
\end{vmatrix}
\]

\[
= (1 - \lambda)(\lambda^2 - 2\lambda + 2)
\]

The first factor gives eigenvalue 1, the second gives eigenvalues 1 ± i.

For \( \lambda = 1 \), we must find the kernel of

\[
\begin{vmatrix}
0 & 2 & -1 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{vmatrix}
\]

which gives the eigenvector

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

For \( \lambda = 1 - i \) we must find the kernel of

\[
\begin{vmatrix}
i & 2 & -1 \\
0 & i & 1 \\
0 & -1 & i
\end{vmatrix}
\]

Solving this we get the eigenvector

\[
\begin{pmatrix}
-2 + i \\
i \\
1
\end{pmatrix}
\]

Taking conjugates, we get that the eigenvector for \( \lambda = 1 + i \) is

\[
\begin{pmatrix}
-2 - i \\
-i \\
1
\end{pmatrix}
\]

9.5.14 Find a general solution to the equation \( x' = Ax \) where
\[
A = \begin{pmatrix}
-1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 3 & -1
\end{pmatrix}
\]

We start by computing the characteristic polynomial.

\[
\det \begin{pmatrix}
-1-\lambda & 1 & 0 \\
1 & 2-\lambda & 1 \\
0 & 3 & -1-\lambda
\end{pmatrix} = -(\lambda^3 - 7\lambda - 6) = -(\lambda + 1)(\lambda + 2)(\lambda - 3)
\]

So the eigenvalues are \(-1, -2, 3\).

For \(\lambda = -1\), we must find the kernel of

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 3 & 1 \\
0 & 3 & 1
\end{pmatrix}
\]

Row reducing we get

\[
\begin{pmatrix}
1 & 3 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

which gives the eigenvector

\[
\begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}
\]

For \(\lambda = -2\), we must find the kernel of

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 4 & 1 \\
0 & 3 & 1
\end{pmatrix}
\]

Row reducing we get

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 3 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

which gives the eigenvector
For $\lambda = 3$, we must find the kernel of
\[
\begin{pmatrix}
-4 & 1 & 0 \\
1 & -1 & 1 \\
0 & 3 & -4
\end{pmatrix}
\]
Row reducing we get
\[
\begin{pmatrix}
1 & -1 & 1 \\
0 & 3 & -4 \\
0 & 0 & 0
\end{pmatrix}
\]
which gives the eigenvector
\[
\begin{pmatrix}
\frac{1}{3} \\
\frac{1}{3} \\
1
\end{pmatrix}
\]
Combining these, we get that the general solution to the differential equation is
\[
c_1 e^{-t} \begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix} + c_2 e^{-2t} \begin{pmatrix}
1 \\
\frac{1}{3} \\
1
\end{pmatrix} + c_3 e^{3t} \begin{pmatrix}
\frac{1}{3} \\
\frac{1}{3} \\
1
\end{pmatrix}
\]
9.5.20 Find a fundamental matrix for the system $x' = Ax$, where
\[
A = \begin{pmatrix}
5 & 4 \\
-1 & 0
\end{pmatrix}
\]
The characteristic polynomial of $A$ is $\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$, so the eigenvalues are $\lambda = 1, 4$. For $\lambda = 1$ we must find the kernel of
\[
\begin{pmatrix}
4 & 4 \\
-1 & -1
\end{pmatrix}
\]
which is spanned by
\[
\begin{pmatrix}
-1 \\
1
\end{pmatrix}
\]
For \( \lambda = 4 \) we must find the kernel of

\[
\begin{pmatrix}
1 & 4 \\
-1 & -4
\end{pmatrix}
\]

which is spanned by

\[
\begin{pmatrix}
-4 \\
1
\end{pmatrix}
\]

The corresponding fundamental matrix is

\[
\begin{pmatrix}
-e^t & -4e^{4t} \\
e^t & e^{4t}
\end{pmatrix}
\]

9.5.26 Find a general solution to the system of equations

\[
\begin{align*}
x' &= 3x - 4y \\
y' &= 4x - 7y
\end{align*}
\]

This system can be rewritten as \( x' = Ax \), where

\[
A = \begin{pmatrix}
3 & -4 \\
4 & -7
\end{pmatrix}
\]

The characteristic polynomial is \( \lambda^2 + 4\lambda - 5 = (\lambda - 1)(\lambda + 5) \), so the eigenvalues are 1 and -5. For \( \lambda = 1 \) we must find the kernel of

\[
\begin{pmatrix}
2 & -4 \\
4 & -8
\end{pmatrix}
\]

which is spanned by

\[
\begin{pmatrix}
2 \\
1
\end{pmatrix}
\]

For \( \lambda = -5 \) we must find the kernel of

\[
\begin{pmatrix}
8 & -4 \\
4 & -2
\end{pmatrix}
\]

which is spanned by
Combining these, we see the general solution to the initial system is \( x = 2c_1 e^t + c_2 e^{-5t}, \ y = c_1 e^t + 2c_2 e^{-5t}. \)

9.5.34 Solve the initial value problem

\[
\begin{align*}
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix} x',
\begin{pmatrix}
-1 \\
4 \\
0 \\
\end{pmatrix}
\end{align*}
\]

From the eigenvectors and eigenvalues from problem 6, the general solution to this equation is

\[
x(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

Plugging in the initial condition, we must solve the equations

\[
\begin{pmatrix}
-1 & -1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
-1 \\
4 \\
0 \\
\end{pmatrix}
\]

Row reducing the system and back-solving gives, \( c_1 = 3, c_2 = -1, c_3 = 1, \) so the desired solution is

\[
x(t) = 3e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\]

9.5.35 a. Show that the matrix

\[
A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}
\]

has a repeated eigenvalue, and only one eigenvector.

The characteristic polynomial is \( \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2, \) so the only eigenvalue is \( \lambda = -1. \) Searching for eigenvectors, we must find the kernel of

\[
\begin{pmatrix}
2 & -1 \\
4 & -2 \\
\end{pmatrix}
\]
which is spanned by

\[
\begin{pmatrix}
1 \\
2
\end{pmatrix}
\]

b. Use your answer to part a. to find a nontrivial solution to \(x' = Ax\).

\[
e^{-t} \begin{pmatrix}
1 \\
2
\end{pmatrix}
\]

c. Try to find a second solution of the form \(te^{-t}u_1 + e^{-t}u_2\).

Plugging this expression into \(x' = Ax\), we get

\[-te^{-t}u_1 + e^{-t}u_1 - e^{-t}u_2 = te^{-t}Au_1 + e^{-t}Au_2.\]

Grouping the \(e^{-t}\) and \(te^{-t}\) terms together, we get to vector relations

\[-u_2 + u_1 = Au_2 \text{ or } (A + I)u_2 = u_1\]

and \(-u_1 = Au_1, \text{ or } (A + I)u_1 = 0.\)

We want \(u_1\) to be an eigenvector. To find \(u_2\), we can either solve the given set of linear equations, or just guess a \(u_2\) and see if \((A + I)u_2\) is an eigenvector. (This may seem ad hoc, but it works as long as your guess for \(u_2\) is not already an eigenvector.) If we guess

\[u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\]

then

\[(A + I)u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}\]

which is an eigenvector. So we get a solution of the differential equation

\[
e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-t} \begin{pmatrix} 2 \\ 4 \end{pmatrix}\]

d. What is \((A + I)^2u_2\)?

\[(A + I)^2u_2 = (A + I)(A + I)u_2 = (A + I)u_1 = 0\]

from the equations we derived in part c.

9.5.36 Use the method of problem 35 to find a general solution to the system
\[
x'(t) = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix} x(t)
\]

Computing the characteristic polynomial, we get that \( \lambda = 2 \) is a double root, and

\[
v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

is an eigenvector, so \( e^{2t} v_1 \) is a solution to the differential equation.

As in problem 35, we guess a solution of the form \( te^{2t} u_1 + e^{2t} u_2 \). This gives rise to the equations \( (A - 2I)u_1 = 0, (A - 2I)^2 u_2 = u_1 \).

Guessing

\[
u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

we get

\[
u_1 = \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}
\]

which is an eigenvector. So another solution is \( te^{2t} u_1 + e^{2t} u_2 \). Combining these, we get a general solution

\[
c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 (te^{2t} \begin{pmatrix} 3 \\ 3 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix})
\]