1. (20 Points) Let $A \in \mathbb{R}^{n \times n}$. Show that $\|A^k\| \leq \|A\|^k$ for all positive integers $k$ and for any induced norm $\| \cdot \|$. 
2. (20 Points) Assume that the matrix $A \in \mathbb{R}^{n \times n}$ is diagonalizable with real eigenvalues. In other words, there exists a diagonal matrix $D$ and a non-singular matrix $T$, with $D, T \in \mathbb{R}^{n \times n}$, such that $A = TDT^{-1}$. Show that

(a) The matrix $TDT^T$ is symmetric.
(b) There exist symmetric matrices $W, V \in \mathbb{R}^{n \times n}$, with $W$ non-singular, so that $A = VW^{-1}$. 
3. (20 Points) Let $W \in \mathbb{R}^{m \times m}$ be a non-singular matrix and let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m}$, with $n \leq m$. Consider the weighted least squares problem $\min_{x} \|W (Ax - b)\|_2$.

(a) Write the normal equation for this problem.

(b) Sketch a backward stable algorithm for solving the problem. Count the number of flops up to the leading terms.
4. (20 Points) Let $A, Q \in \mathbb{R}^{n \times n}$ with $Q = (q_1 \cdots q_n)$ orthogonal. Suppose that

$$Q^T AQ = H$$

is upper Hessenberg such that all subdiagonal entries of $H$ are positive. The Implicit $Q$ Theorem states that columns $q_2$ through $q_n$ are uniquely determined by $q_1$. Derive the Arnoldi algorithm from (1).
5. (20 Points) Use the SVD to show that if $A \in \mathbb{R}^{m \times n}$ with $m \geq n$, then there exists $Q \in \mathbb{R}^{m \times n}$ with orthonormal columns ($Q^TQ = I$) and a positive semidefinite matrix $P \in \mathbb{R}^{n \times n}$ such that $A = QP$. 