Solution to Problem 6.5: We can write

\[ A = X\Lambda_A X^{-1}, \quad B = Y\Lambda_B Y^{-1}. \]

By Lemma 6.3,

\[
I \otimes A + B \otimes I = I \otimes (X\Lambda_A X^{-1}) + (Y\Lambda_B Y^{-1}) \otimes I \\
= (Y^{-1} \otimes X) \Lambda_A (Y \otimes X) + (Y \Lambda_B Y^{-1}) \otimes (X X^{-1}) \\
= (Y \otimes X) (I \otimes \Lambda_A + \Lambda_B \otimes I) (Y^{-1} \otimes X^{-1}) \\
= (Y \otimes X) (I \otimes \Lambda_A + \Lambda_B \otimes I) (Y \otimes X) (Y \otimes X)^{-1}.
\]

The matrix \( I \otimes \Lambda_A + \Lambda_B \otimes I \) is a diagonal matrix, with its diagonal entries being \( \alpha_i + \beta_j \) for \( 1 \leq i, j \leq n \).

The Sylvester equation \( AX + XB^T = C \) can be equivalently written as

\[
(I \otimes A + B \otimes I) \text{vec}(X) = \text{vec}(C).
\]

This equation is solvable for any given \( C \) if and only if \( I \otimes A + B \otimes I \) is non-singular. In other words, if and only if \( \alpha_i + \beta_j \neq 0 \) for all \( 1 \leq i, j \leq n \). Since \( B \) and \( B^T \) have the same set of eigenvalues, this is the same set of conditions for the non-singularity of the alternative Sylvester equation \( AX + XB = C \).