Math221: Matrix Computations

Homework #8 Solutions

- Problem 4.6:

1. Let \( A = Q_A A' Q_A^* \) and \( B = Q_B B' Q_B^* \) be the Schur decompositions of \( A \) and \( B \), respectively. Both \( A' \) and \( B' \) are upper triangular and \( Q_A \) and \( Q_B \) are unitary. It follows that

\[
Q_A A' Q_A^* X - X Q_B B' Q_B^* = C,
\]

and that

\[
A' Y - Y B' = C',
\]

where \( Y = Q_A^* X Q_B \) and \( C' = Q_A^* C Q_B \). Once we have solved for \( Y \), we can recover \( X \) by computing \( X = Q_A Y Q_B^* \).

2. Partition

\[
A' = \begin{pmatrix} \tilde{A} & a \\ \alpha & \end{pmatrix}, \quad B' = \begin{pmatrix} \tilde{B} & b \\ \beta & \end{pmatrix} \quad \text{and} \quad C' = \begin{pmatrix} \tilde{C} & c_1 \\ c_2^* & \gamma \end{pmatrix}
\]

and

\[
Y = \begin{pmatrix} \tilde{Y} & y_1 \\ y_2^* & \delta \end{pmatrix}.
\]

It follows from (1) that

\[
\begin{align*}
\alpha y_2^* - y_2^* \tilde{B} &= c_2^* \\
(\alpha - \beta) \delta - y_2^* b &= \gamma \\
\tilde{A} y_1 + \delta a - \tilde{Y} b - \beta y_1 &= c_1 \\
\tilde{A} \tilde{Y} + a y_2^* - \tilde{Y} \tilde{B} &= \tilde{C}.
\end{align*}
\]

Equation (2) implies

\[
y_2^* \left( \alpha I - \tilde{B} \right) = c_2^*.
\]

which has a unique solution for any \( c_2 \) when all eigenvalues of \( \tilde{B} \) differ from \( \alpha \). Having computed \( y_2^* \), we can compute \( \delta \) from equation (3) as

\[
(\alpha - \beta) \delta = y_2^* b + \gamma.
\]
This equation has a unique solution when $\alpha$ differs from $\beta$. Overall, we can solve both $y_2$ and $\delta$ uniquely as long as $\alpha$ is not an eigenvalue of $B'$.

From equation (5), we can now proceed to recursively compute the lower triangular part of $Y$ from the following equation:

$$\tilde{A}Y - \tilde{Y}B = \tilde{C} - ay_2^*.$$ 

As long as $A'$ and $B'$ do not share eigenvalues, we can proceed to uniquely determine the lower triangular part of $Y$.

Having done so, we can now further determine the strictly upper triangular part of $Y$, one column at a time, from left to right. With an induction argument, assuming we have determined all upper triangular components of $\tilde{Y}$, so that all components of $\tilde{Y}$ have been determined, we can now proceed to determine $y_1$ from equation (4) as

$$\left(\tilde{A} - \beta I\right)y_1 = c_1 - \delta a + \tilde{Y}b.$$ 

This equation has a unique solution when $A'$ and $B'$ do not share a common eigenvalue.

To recap, we solve the lower triangular part of $Y$ from right to left, one column at a time, including the diagonals. Once this is done, we solve for the strictly upper triangular part of $Y$ from left to right, one column at a time. (Alternatively, we can also solve for $Y$ from top left to bottom right, one column and one row at a time.)

• Problem 4.7: Since $S^{-1} = \begin{pmatrix} I & -R \\ R & I \end{pmatrix}$, we have

$$S^{-1}TS = \begin{pmatrix} I & -R \\ R & I \end{pmatrix} \begin{pmatrix} A & C \\ B & I \end{pmatrix} \begin{pmatrix} I & R \\ I & C + AR - RB \end{pmatrix}.$$ 

Hence we can choose $R$ such that

$$R B - A R = C.$$ 

In order for this equation to indeed have a solution, we require that $A$ and $B$ have no common eigenvalues.

• Problem 4.8: Let $X = \begin{pmatrix} I & -A \\ I & I \end{pmatrix}$. Then $X^{-1} = \begin{pmatrix} I & A \\ I & I \end{pmatrix}$ and

$$X \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} X^{-1} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}.$$ 

Since eigenvalues of $\begin{pmatrix} AB & A \\ B & 0 \end{pmatrix}$ are those of $AB$ and $0$’s, and eigenvalues of $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ are those of $BA$ and $0$, and since these matrices are similar, it follows that both $AB$ and $BA$ have the same set of non-zero eigenvalues.

• Problem 4.10:
1. $H = (A + A^*)/2$ is Hermitian, whereas $S = (A - A^*)/2$ is skew-Hermitian, with $A = H + S$. This decomposition is unique.

2. Let $A = QUQ^*$ be the Schur form for $A$, where $Q$ is unitary and $U$ upper triangular. Then the eigenvalues of $A$ are simply the diagonal entries of $U$. Let $U = \tilde{H} + \tilde{S}$, where $\tilde{H}$ Hermitian and $\tilde{S}$ skew-Hermitian. It follows that the diagonal entries on the main diagonal of $\tilde{H}$ are the real parts of the eigenvalues of $A$. Consequently,

$$
\sum_i |\Re(\lambda_i)|^2 \leq \|\tilde{H}\|_F^2.
$$

Since

$$
\tilde{H} = (U + U^*)/2 = (Q^*AQ + (Q^*AQ)^*)/2 = Q^*HQ,
$$

it follows that

$$
\|\tilde{H}\|_F = \|H\|_F.
$$

Hence

$$
\sum_i |\Re(\lambda_i)|^2 \leq \|H\|_F^2.
$$

3. Continue the arguments above, since and the diagonal entries on the main diagonal of $\tilde{S}$ are the imaginary parts of the eigenvalues of $A$, and since $\|\tilde{S}\|_F = \|S\|_F$, we have

$$
\sum_i |\Im(\lambda_i)|^2 \leq \|S\|_F^2.
$$

4. Again let $A = QUQ^*$ be the Schur form for $A$. Then we have to prove that $A$ is normal if and only $U$ is. So we have to prove that $A$ is normal if and only if

$$
\sum_i |\Re(\lambda_i)|^2 = \|U\|_F^2.
$$

On the other hand, since $\Re(\lambda_i)$ are the eigenvalues of $A$, they must be on the diagonal of $U$. The last equation therefore implies that all off-diagonal entries of $U$ must vanish. Hence $U$ is diagonal and hence $A = QUQ^*$ must be normal.