Math221: Matrix Computations

Homework #3 Solutions

• 2.13 (3): Define $y_0 = c$ and
  
  $$y_{k+1} = y_k - A^{-1} (B y_k - c), \quad k = 0, 1, 2, \ldots.$$

  Then
  
  $$y_{k+1} - B^{-1} c = \left( I - A^{-1} B \right) \left( y_k - B^{-1} c \right).$$

  Hence
  
  $$\| y_{k+1} - B^{-1} c \| \leq \| A^{-1} \| \| A - B \| \| y_k - B^{-1} c \|.$$

  For $\| A - B \|$ sufficiently small, $\| A^{-1} \| \| A - B \| < 1$ and hence the limit of the sequence $\{ y_k \}$ is $B^{-1} c$.

• 2.18: We will assume that all the leading principal submatrices of $A$ are non-singular. If this is not the case, a simple continuity argument would make up for the gap left by this assumption.

  Assume that we have performed $k$ steps of Gaussian elimination, so that
  
  $$A = \begin{pmatrix} L_{11} & I \\ L_{21} & I \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ \hat{S} \end{pmatrix},$$

  where $\hat{S}$ is the matrix that overwrites $A_{22}$.

  On the other hand, direct block elimination also gives
  
  $$A = \begin{pmatrix} I \\ A_{21} A_{11}^{-1} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ \hat{S} \end{pmatrix}.$$

  Replacing $A_{11}$ by its LU factorization $A_{11} = L_{11} U_{11}$, and by the uniqueness of the LU factorization, we can rewrite the above equation as
  
  $$A = \begin{pmatrix} L_{11} \\ L_{21} & I \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ \hat{S} \end{pmatrix}.$$

  Hence $\hat{S} = S$. 
• Problems 2.20:

  – (a): Compute GEPP $A = PLU$. Solving $A^k x = b$ then involves $k$ permutations as well as $k$ forward and backward substitutions. Total cost: $2/3n^3 + kn^2 + O(n^2)$.

  – (b): Compute GEPP $A = PLU$, and solve for $A^{-1} b$.

  – (c): Compute GEPP $A = PLU$. Solving $AX = B$ then involves $m$ permutations as well as $m$ forward and backward substitutions. Total cost: $2/3n^3 + mn^2 + O(n^2)$. 