How Gauss Determined the Orbit of Ceres

Math 221
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Introduction

• Giuseppe Piazzi: discovered Ceres on Jan. 1, 1801
  – Made 19 observations over 42 days
  – Then, object was lost in glare of the Sun

<table>
<thead>
<tr>
<th>Date</th>
<th>Right Ascension</th>
<th>Declination</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan. 2</td>
<td>51° 47’ 49”</td>
<td>15° 41’ 5”</td>
<td>8 h 39 min 4.6 sec</td>
</tr>
<tr>
<td>Jan. 22</td>
<td>51° 42’ 21”</td>
<td>17° 3’ 18”</td>
<td>7 h 20 min 21.7 sec</td>
</tr>
<tr>
<td>Feb. 11</td>
<td>54° 10’ 23”</td>
<td>18° 47’ 59”</td>
<td>6 h 11 min 58.2 sec</td>
</tr>
</tbody>
</table>

• Carl Gauss: calculated the orbit of Ceres
  – Originally used only 3 of Piazzi’s observations
  – Initiated the theory of least squares
The orbit of Ceres is determined by six quantities: \( i, \Omega, \pi, a, e, \tau \)
Gauss’ method using 3 points

Piazzi’s data: lines of sight $L_1$, $L_2$, $L_3$ and elapsed times between observations

Sectoral areas swept out by orbit are proportional to elapsed times

Approximate sectoral areas with triangular areas

\[
\frac{T_{23}}{T_{13}} = \text{(approximately)} \frac{S_{23}}{S_{13}} = 0.513, \quad \text{"c"}
\]

\[
\frac{T_{12}}{T_{23}} = \text{(approximately)} \frac{S_{12}}{S_{23}} = 0.487, \quad \text{"d"}
\]

\[
\frac{S_{12}}{S_{23}} = \frac{t_2-t_1}{t_3-t_2} = 0.94952
\]

\[
\frac{S_{12}}{S_{13}} = \frac{t_2-t_1}{t_3-t_1} = 0.48705
\]

\[
\frac{S_{23}}{S_{13}} = \frac{t_3-t_2}{t_3-t_1} = 0.51295
\]

12/7/2007

Orbit of Ceres
Gauss’ method using 3 points

Determine the point F in the plane of earth’s orbit

First, find points F1 and F3

Use principle of parallel displacements to find point F

Length’s OE1 and OE3 are known. We find lengths OF1 and OF3 with

\[
\frac{OF_1}{OE_1} = c \quad \text{and} \quad \frac{OF_3}{OE_3} = d
\]
Gauss' method using 3 points

Draw lines $L_1'$ and $L_3'$ parallel to $L_1$ and $L_3$, passing through $F$. This defines a unique plane $Q$. Where plane $Q$ intersects $L_2$ is the point $P_2$.

However, the area $T_{13}$ is much different than $S_{13}$.

Gauss' correction factor

\[
\frac{S_{13}}{T_{13}} = 1 + \left( 2 \times \frac{\pi^2 \times (r_2 - r_1) \times (r_3 - r_2)}{r_2^3} \right) = G
\]

\[
\frac{T_{12}}{T_{13}} = G \times \frac{S_{12}}{S_{13}} = G \times \frac{r_2 - r_1}{r_3 - r_1}, \quad \frac{T_{23}}{T_{13}} = G \times \frac{r_3 - r_2}{r_3 - r_1}.
\]

Iterate: let $G=1$, then calculate $r_2$, calculate $G$, recalculate $r_2$, etc…
Gauss’ method using 3 points
Finding the other two points $P_1$ and $P_3$.

\[
\frac{FQ_1'}{FP_1'} = \frac{T_{23}}{T_{13}}
\]
\[
\frac{FQ_3'}{FP_3'} = \frac{T_{12}}{T_{13}}
\]

$FP_1' = E_1P_1$
$FP_3' = E_3P_3$
Setting up the equations

-> the goal is to determine the distance Sun-Ceres \( r_1, r_2, r_3 \), and deduce others quantities from it

In his initial paper, Gauss first set up 16 equations involving \( r_1, r_2, r_3 \) and the area of the triangle \( T_{12}, T_{23} \) and \( T_{13} \).
Those equations are reduced to 4 by considering geometric identity: non-linear equations

\[
(F + F'')(f' r_2 [\pi \pi' \pi''] = (F f' - F'' f)(D[\pi P \pi''] - D'[\pi P' \pi'']) + (F'(f + f'') - (F + F'')f')D'\pi P' \pi'"
\]

(1)

\[
(F + F')(f' r_2 [\pi \pi' P'] + f'' r_3 [\pi \pi'' P']) = (F f'' - F'' f)(D[\pi PP'] - D''[\pi P'' P'])
\]

(2)

\[
(F - F'')(f r_1 [\pi' \pi P] + f'' r_3 [\pi' \pi'' P']) = (F f'' - F'' f)(D[\pi' PP'] - D''[\pi' P'' P'])
\]

(3)

\[
(F + F')(f r_1 [\pi'' \pi P] + f'' r_2 [\pi'' \pi' P']) = (F f'' - F'' f)(D[\pi'' PP'] - D''[\pi'' P'' P'])
\]

(4)

If we consider \( f' = T_{13} \approx S_{13}, f = T_{23} \approx S_{23}, f'' = T_{12} \approx S_{12} \) there are four equations for 3 unknowns.
In practice, Gauss didn’t use the third equation
Solving the equations

In equation 2 and 4, Gauss build an approximation by removing terms of order $O(t^7)$ This way, we can express $r_1$ and $r_3$ in term of $r_2$.

\[ r_1 = \frac{g}{f'} \cdot \frac{f'' - \tau}{\tau'' - \tau'} \cdot \left[ \pi \pi' P' \right] r_2 \]

\[ r_3 = \frac{g''}{f''} \cdot \frac{f' - \tau}{\tau' - \tau} \cdot \left[ \pi \pi' P' \right] r_2 \]

Apparently in his earliest work, Gauss approximate $f' = T_{13} \approx S_{13}$, $f = T_{23} \approx S_{23}$, $f'' = T_{12} \approx S_{12}$

With some approximation on $f$, $f'$ and $f''$ and using the equation 1: Gauss found a non-linear equation involving only $r_2$

\[ \frac{R'}{r'} = \frac{R'}{r_2} \sqrt{1 + tan^2 \beta' + \left( \frac{R'}{r_2} \right)^2 + 2 \frac{R'}{r_2} \cos (\lambda' - \xi')} \quad \left( 1 - \left( \frac{R'}{r'} \right)^3 \right) \frac{R'}{r_2} = M \]

Very few information about his method to solve this equation
Solving the equations

Extract from Gauss’ book

153.

In the second hypothesis we shall assign to \( P, Q \), the very values, which in the first we have found for \( P', Q' \). We shall put, therefore,

\[
\begin{align*}
x &= \log P = 0.0790168 \\
y &= \log Q = 8.5476110
\end{align*}
\]

Since the calculation is to be conducted in precisely the same manner as in the first hypothesis, it will be sufficient to set down here its principal results:

\[
\begin{align*}
\omega &= 13^\circ 15' 38'' .18 \\
\omega + \sigma &= 13^\circ 38' 51.25 \\
\log Q \sin \omega &= 0.5939989 \\
\log r' &= 0.3253001 \\
\log \frac{r'}{u} &= 0.6675133 \\
\log \frac{d'r'}{du} &= 0.5836029 \\
\xi &= 205^\circ 16' 38'.16
\end{align*}
\]

It would hardly be worth while to compute anew the reductions of the times on account of aberration, for they scarcely differ \( \frac{1}{1000} \) from those which we have got in the first hypothesis.

The further calculations furnish \( \log \eta = 0.0002270 \), \( \log \eta'' = 0.0003173 \), whence are derived

\[
\begin{align*}
\log P' &= 0.0790167 \\
\log Q' &= 8.5476110
\end{align*}
\]

From this it appears how much more exact the second hypothesis is than the first.
Using more data points

• For 3 points fix 2 and look at error in the calculation for the 3\textsuperscript{rd}
• For 4 points fix 2 and look at total error in the calculation for the other 2
• In general, can fix 2 points and look at the error in the calculation for the remaining points, i.e. sum of squares
\[ \sum_{i} e_{i}^{2} \]
Minimizing the Error

• Minimize error

\[ \nabla \left( \sum_i e_i^2 \right) = \sum_i 2e_i \nabla e_i = 0 \]

• Difficult to solve for nonlinear problems, e.g., finding the orbit of Ceres
Linear Problems

- For linear problems
  
  \[ e_i = r_i = (Ax - b)_i \]
  
  \[ \sum_i e_i^2 = \|r\|_2^2 = \|Ax - b\|_2^2 \]

- Want to solve

  \[ \nabla \left( \sum_i e_i^2 \right) = \nabla (\|Ax - b\|_2^2) = 2(Ax - b)^t A = 0 \]

  \[ \Leftrightarrow \quad A^tAx - A^tb = 0 \]
Conclusions

• Gauss’ method evolved over time
• Initially used only 3 points
• Ambiguous whether Gauss applied theory of least squares to Ceres
• Theory of matrix computations was still being developed as Gauss created his method
References

• Tennenbaum, J and Director, B. “How Gauss Determined the Orbit of Ceres.”
• Gauss, C. “Summary Overview of the Method Which was Applied to the Determination of the Orbits of the Two New Planets.”