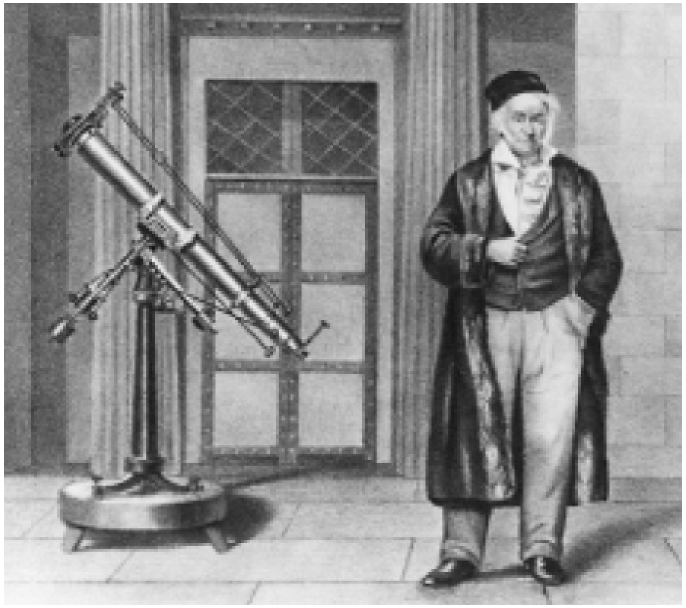
How Gauss Determined the Orbit of Ceres



Math 221

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Introduction

- Giuseppe Piazzi: discovered Ceres on Jan. 1, 1801
  - Made 19 observations over 42 days
  - Then, object was lost in glare of the Sun

	right ascension	declination	Time
Jan. 2	51° 47′ 49″	15° 41′ 5″	8 h 39 min 4.6 sec
Jan. 22	51° 42′ 21″	17° 3′ 18″	7 h 20 min 21.7 sec
Feb. 11	54° 10′ 23″	18° 47′ 59″	6 h 11 min 58.2 sec

- Carl Gauss: calculated the orbit of Ceres
  - Originally used only 3 of Piazzi's observations
  - Initiated the theory of least squares

Orbital characteristics

The orbit of Ceres is determined by six quantities: i, Ω, π, a, e, τ

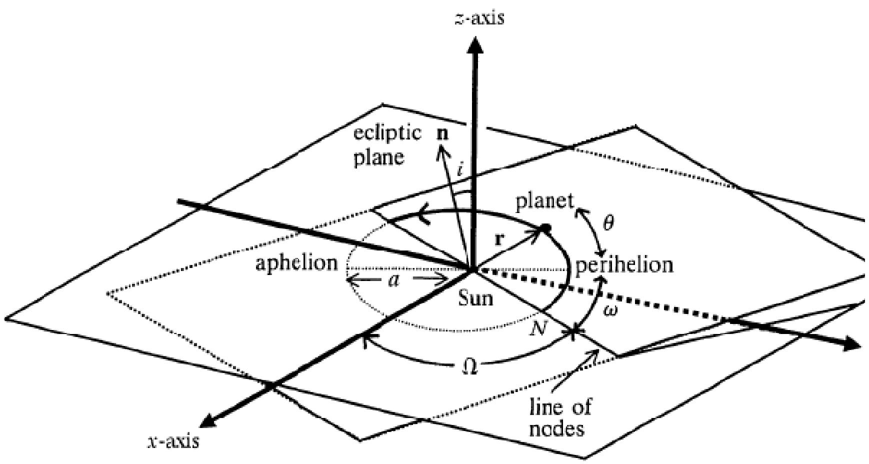
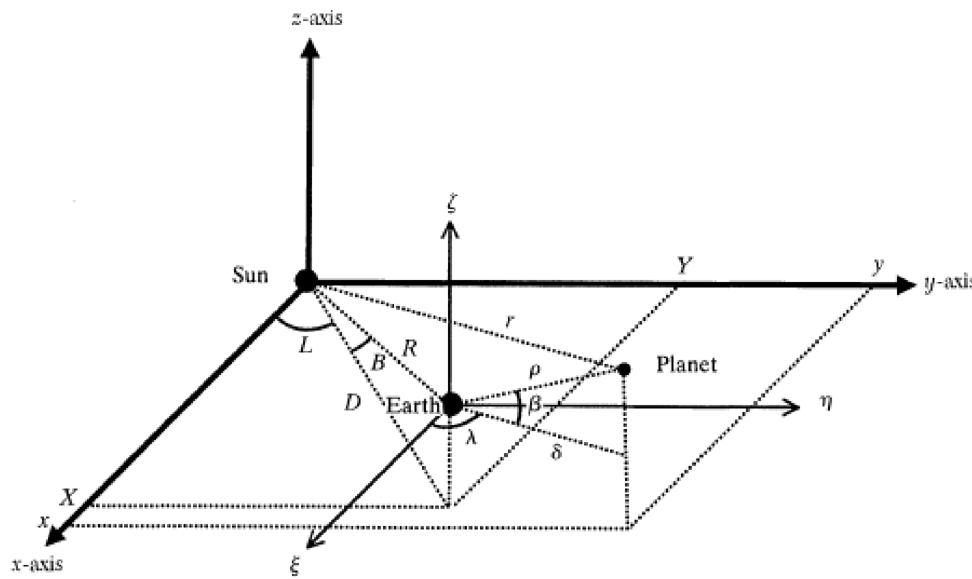
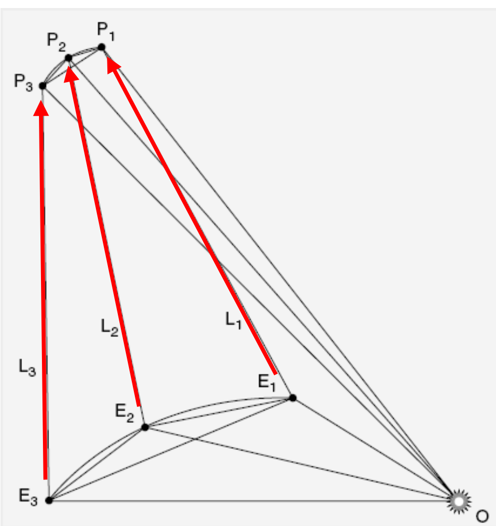


FIGURE 1
Parameters describing the planetary orbit

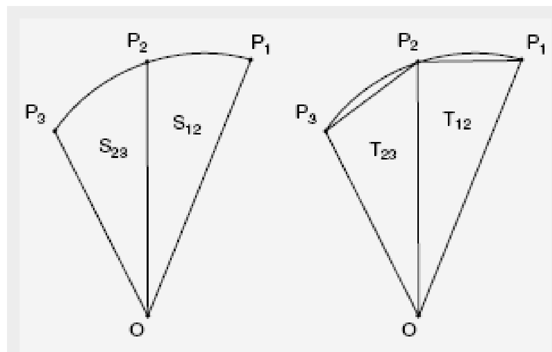


Gauss' method using 3 points

Piazzi's data: lines of sight L_1, L_2, L_3 and elapsed times between observations



Sectoral areas swept out by orbit are proportional to elasped times



$$\frac{S_{12}}{S_{23}} = \frac{t_2 - t_1}{t_3 - t_2} = 0.94952\,,$$

$$\frac{S_{12}}{S_{13}} = \frac{t_2 - t_1}{t_3 - t_1} = 0.48705\,,$$

$$\frac{S_{23}}{S_{13}} = \frac{t_3 - t_2}{t_3 - t_1} = 0.51295\,.$$

Approximate sectoral areas with triangular areas

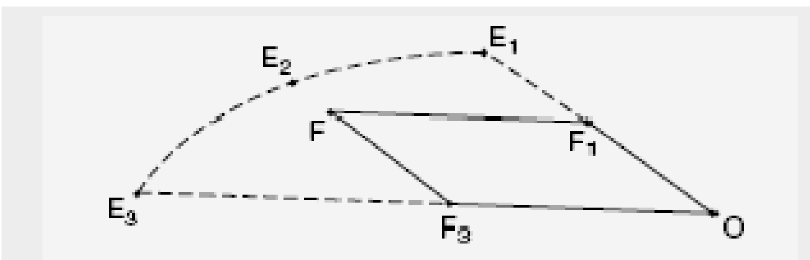
$$\frac{T_{23}}{T_{13}} = \text{(approximately)}\ \frac{S_{23}}{S_{13}} = 0.513\,, \qquad =\text{“c”}$$

$$\frac{T_{12}}{T_{23}} = \text{(approximately)}\ \frac{S_{12}}{S_{23}} = 0.487\,. \qquad =\text{“d”}$$

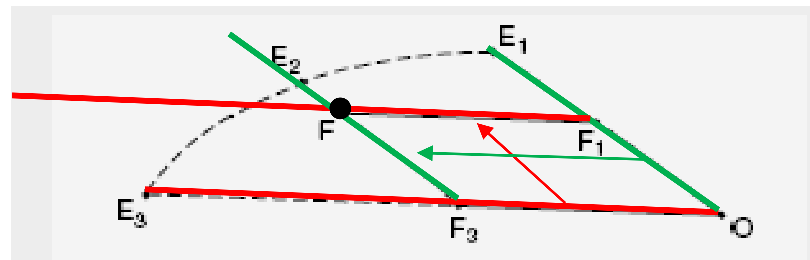
Gauss' method using 3 points

Determine the point F in the plane of earth's orbit

First, find points F1 and F3



Use principle of parallel displacements to find point F

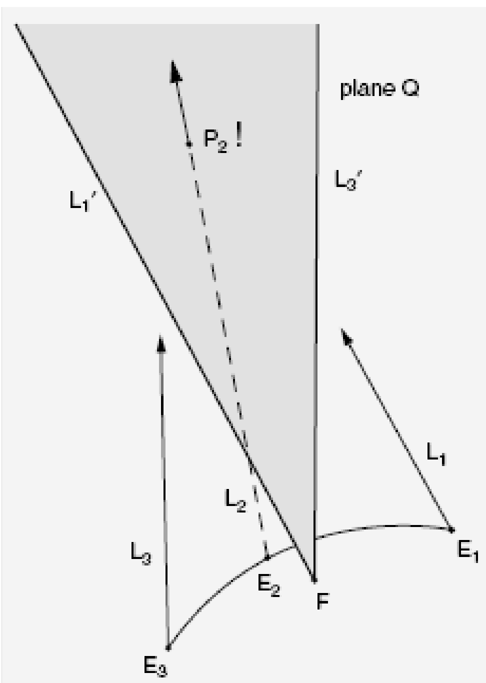


Length's OE_1 and OE_3 are known. We find lengths OF_1 and OF_3 with

$$OF_1/OE_1 = c \text{ and } OF_3/OE_3 = d$$

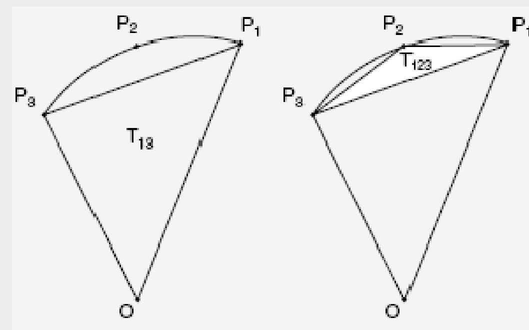
Gauss' method using 3 points

Draw lines L_1' and L_3' parallel to L_1 and L_3, passing through F. This defines a unique plane Q. Where plane Q intersects L_2 is the point P_2.



However, the area T_{13} is much different than S_{13}

FIGURE 13.4. *Most of the excess of S_{13} over T_{13}, which is the lune-shaped area, is constituted by triangle T_{123}.*



Gauss' correction factor

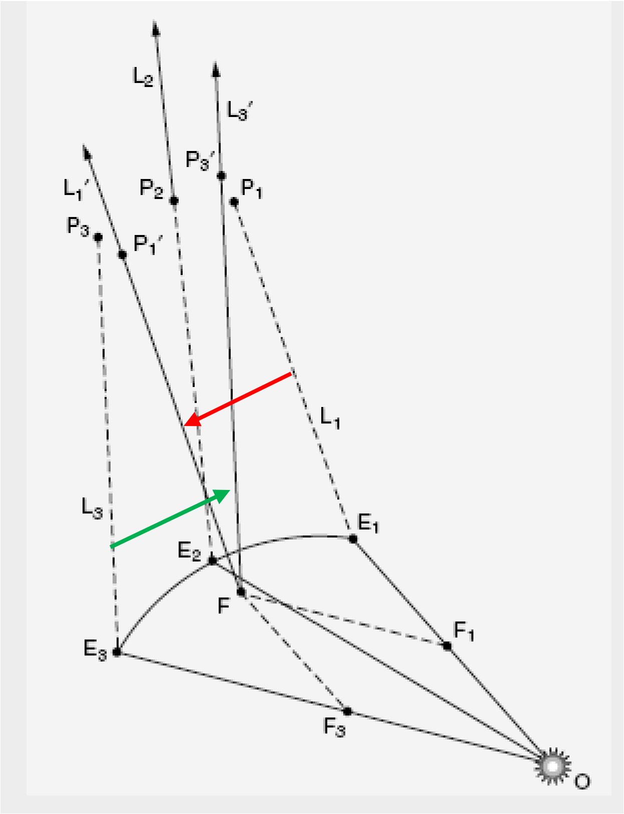
$$\frac{S_{13}}{T_{13}} \simeq 1 + \left(2 \times \frac{\pi^2 \times (t_2 - t_1) \times (t_3 - t_2)}{r_2^3}\right) . \quad = G$$

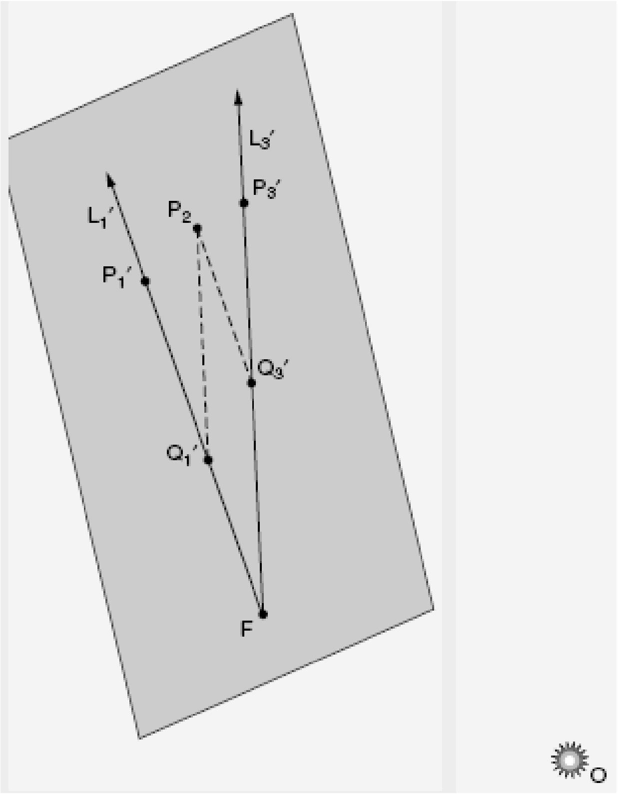
$$\frac{T_{12}}{T_{13}} \simeq G \times \frac{S_{12}}{S_{13}} = G \times \frac{t_2 - t_1}{t_3 - t_1} . \qquad \frac{T_{23}}{T_{13}} \simeq G \times \frac{t_3 - t_2}{t_3 - t_1} .$$

Iterate: let G=1, then calculate r_2, calculate G, recalculate r_2, etc…

Gauss' method using 3 points

Finding the other two points P_1 and P_3.





$$\frac{FQ_1'}{FP_1'} = \frac{T_{23}}{T_{13}}$$

$$\frac{FQ_3'}{FP_3'} = \frac{T_{12}}{T_{13}}$$

$FP_1' = E_1P_1$

$FP_3' = E_3P_3$

Setting up the equations

-> the goal is to determine the distance Sun-Ceres r_1, r_2, r_3 , and deduce others quantities from it

In his initial paper, Gauss first set up 16 equations involving r_1, r_2, r_3 and the area of the triangle T_{12}, T_{23} and T_{13}.

Those equations are reduced to 4 by considering geometric identity: non-linear equations

$$
\begin{aligned}
(F+F'')f'r_2[\pi\pi'\pi''] = (Ff'-F''f)(D[\pi P\pi'']-D''[\pi P''\pi'']) &\quad + \quad (F'(f+f'')-(F+F'')f')D'[\pi P'\pi''] && (1)\\
(F+F'')(f'r_2[\pi\pi'P']+f''r_3[\pi\pi''P']) &\quad = \quad (Ff''-F''f)(D[\pi PP']-D''[\pi P''P']) && (2)\\
(F-F'')(fr_1[\pi'\pi P]+f''r_3[\pi'\pi''P']) &\quad = \quad (Ff''-F''f)(D[\pi'PP']-D''[\pi'P''P']) && (3)\\
(F+F'')(fr_1[\pi''\pi P]+f''r_2[\pi''\pi'P']) &\quad = \quad (Ff''-F''f)(D[\pi''PP']-D''[\pi''P''P']) && (4)
\end{aligned}
$$

If we consider $f' = T_{13} \approx S_{13},\ f = T_{23} \approx S_{23},\ f'' = T_{12} \approx S_{12}$ there are four equations for 3unknowns.
In practice, Gauss didn't use the third equation

Solving the equations

In equation 2 and 4, Gauss build an approximation by removing terms of order $O(t^7)$
This way, we can express r_1 and r_3 in term of r_2.

$$r_1 = \frac{g}{f} \cdot \frac{f'}{g'} \cdot \frac{\tau'' - \tau}{\tau'' - \tau'} \cdot \frac{[\pi'\pi''P']}{[\pi\pi''P']} r_2$$

$$r_3 = \frac{g''}{f''} \cdot \frac{f'}{g'} \cdot \frac{\tau'' - \tau}{\tau' - \tau} \cdot \frac{[\pi\pi'P']}{[\pi\pi''P']} r_2$$

Apparently in his earliest work, Gauss approximate

$$f' = T_{13} \approx S_{13},\ f = T_{23} \approx S_{23},\ f'' = T_{12} \approx S_{12}$$

With some approximation on f, f' and f'' and using the equation 1:
Gauss found a non-linear equation involving only r_2

$$\frac{R'}{r'} = \frac{R'}{r_2}\sqrt{1 + tan^2\beta' + \left(\frac{R'}{r_2}\right)^2 + 2\frac{R'}{r_2}cos(\lambda' - L')}$$

$$\left(1 - \left(\frac{R'}{r'}\right)^3\right)\frac{R'}{r_2} = M$$

Very few information about his method to solve this equation

Solving the equations

Extract from Gauss' book

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In the *second hypothesis* we shall assign to P, Q, the very values, which in the first we have found for P', Q'. We shall put, therefore,

$$x = \log P = 0.0790164$$
$$y = \log Q = 8.5475981$$

Since the calculation is to be conducted in precisely the same manner as in the first hypothesis, it will be sufficient to set down here its principal results:—

ω	13°15′38″.13	ζ''	210° 8′24″.98
$\omega + \sigma$	13 38 51 .25	$\log r$	0.3307676
$\log Q c \sin \omega$	0.5989389	$\log r''$	0.3222280
z	14 33 19 .00	$\frac{1}{2}(u'' + u)$	205 22 15 .58
$\log r'$	0.3259918	$\frac{1}{2}(u'' - u)$	—3 14 4 .79
$\log \frac{n' r'}{n}$	0.6675193	$2f'$	7 34 53 .32
$\log \frac{n' r'}{n''}$	0.5885029	$2f$	3 29 0 .18
ζ	203 16 38 .16	$2f''$	4 5 53 .12

It would hardly be worth while to compute anew the reductions of the times on account of aberration, for they scarcely differ 1ˢ from those which we have got in the first hypothesis.

The further calculations furnish $\log \eta = 0.0002270$, $\log \eta'' = 0.0003173$, whence are derived

$$\log P' = 0.0790167 \qquad X = +0.0000003$$
$$\log Q' = 8.5476110 \qquad Y = +0.0000129$$

From this it appears how much more exact the second hypothesis is than the first.

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Using more data points

- For 3 points fix 2 and look at error in the calculation for the 3rd
- For 4 points fix 2 and look at total error in the calculation for the other 2
- In general, can fix 2 points and look at the error in the calculation for the remaining points, i.e. sum of squares

$$\sum_i e_i^2$$

Minimizing the Error

- Minimize error

$$\nabla\left(\sum_i e_i^2\right) = \sum_i 2e_i \nabla e_i = 0$$

- Difficult to solve for nonlinear problems, e.g., finding the orbit of Ceres

Linear Problems

- For linear problems

$$e_i = r_i = (Ax - b)_i$$

$$\sum_i e_i^2 = ||r||_2^2 = ||Ax - b||_2^2$$

- Want to solve

$$\nabla(\sum_i e_i^2) = \nabla(||Ax - b||_2^2) = 2(Ax - b)^t A = 0$$

$$\Leftrightarrow \quad A^t A x - A^t b = 0$$

Conclusions

- Gauss’ method evolved over time
- Initially used only 3 points
- Ambiguous whether Gauss applied theory of least squares to Ceres
- Theory of matrix computations was still being developed as Gauss created his method

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