Homework 3 Math 170  
Reader: Michael Pejic mpejic@math.berkeley.edu

**Problem set:**  *Franklin* Section 1.5, Problems 1,2,4,5,10; Section 1.4, Problems 1,2,3,5  

**Problems checked:**  *Franklin* Section 1.3, Problems 1,2; Section 1.4, Problems 1,2,3  

**Grading scheme:**

- $X$ for “complete”: significant effort demonstrated  
- $O$ for “fail”: lack of demonstration of significant effort  

**Problems graded:**  *Franklin Franklin* Section 1.5, Problems 4,5,10; Section 1.6, Problem 5  

**Grading scheme:**

- 3 for “excellent”: Necessary steps are all shown and well explained.  
  Solution is correct.  
- 2 for “fair”: Necessary steps are all shown.  
  There are minor gaps in explanation and/or minor errors in solution.  
- 1 for “poor”: Necessary steps are lacking.  
  There are major gaps in explanation and/or major errors in solution.  
- 0 for “fail”: Significant effort is not demonstrated.
Sample solutions:

**Franklin Section 1.5, Problem 4** The three basic solutions are given by
\[
\begin{pmatrix}
-1 & 1 & 1 & 0 \\
2 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
z
\end{pmatrix} =
\begin{pmatrix}
1 \\
1
\end{pmatrix}, \text{x} \geq 0, z \geq 0, \text{minimize } z_1 + z_2
\]

Starting with the basic feasible solution
\[
\begin{pmatrix}
0 \\
0 \\
1 \\
1
\end{pmatrix}
\]
with \( M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M^{-1} \), a Phase II calculation involving the second column of the matrix gives
\[
\begin{pmatrix}
t_4 \\
t_5
\end{pmatrix} =
M^{-1}
\begin{pmatrix}
0 \\
1
\end{pmatrix}, p = 5, \lambda^* = 1, \text{leading to a new feasible solution }
\begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}
\]

with \( M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M^{-1} \). Another Phase II calculation involving the third column of the matrix gives
\[
\begin{pmatrix}
t_2 \\
t_4
\end{pmatrix} = M^{-1}
\begin{pmatrix}
1 \\
0
\end{pmatrix} =
\begin{pmatrix}
0 \\
1
\end{pmatrix}, p = 4, \lambda^* = 1, \text{leading to a new feasible solution }
\begin{pmatrix}
0 \\
1 \\
1 \\
0
\end{pmatrix}
\]

which corresponds to the feasible solution
\[
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
\]
in the first step.

Alternatively, starting with the same basic feasible solution, if instead a Phase II calculation involving the first column of the matrix were done, it would give
\[
\begin{pmatrix}
t_4 \\
t_5
\end{pmatrix} = M^{-1}
\begin{pmatrix}
-1 \\
1
\end{pmatrix} =
\begin{pmatrix}
-1 \\
1
\end{pmatrix}, p = 5, \lambda^* = 1, \text{leading to a new
feasible solution \[
\begin{bmatrix}
1 \\
0 \\
2 \\
0
\end{bmatrix}
\] with \( M = \begin{bmatrix} -1 & 1 \\
1 & 0 \end{bmatrix} \) \(\Leftrightarrow\) \( M^{-1} = \begin{bmatrix} 0 & 1 \\
1 & 1 \end{bmatrix} \). Another Phase II calculation involving the third column of the matrix gives \[
\begin{bmatrix}
t_1 \\
t_4
\end{bmatrix} =
M^{-1} \begin{bmatrix} 1 \\
0
\end{bmatrix} = \begin{bmatrix} 1 \\
0
\end{bmatrix}, \]
\( p = 4, \lambda^* = 2 \), leading to a new feasible solution \[
\begin{bmatrix} 1 \\
0 \\
2 \\
0
\end{bmatrix},
\]
which corresponds to the feasible solution \[
\begin{bmatrix} 1 \\
0 \\
2 \\
0
\end{bmatrix}
\] in the first step.

**Franklin Section 1.5, Problem 5** Clearly the feasible basic solution \[
\begin{bmatrix} 0 \\
1 \\
1
\end{bmatrix}
\] minimizes \( x_1 + x_2 + x_3 \). To confirm this result using a Phase II calculation, start with this feasible solution and perform a Phase II calculation involving the first column of the matrix. \( M = \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix} = M^{-1} \), so \[
\begin{bmatrix}
t_2 \\
t_3
\end{bmatrix} M^{-1} \begin{bmatrix} -1 \\
1
\end{bmatrix} = \begin{bmatrix} 1 \\
-1
\end{bmatrix}.\]
Since \( c_1 = c_2 = c_3 = 1 \), so \( t_2 c_2 + t_3 c_3 - c_1 = -1 < 0 \), we are in Case 1 and our basic feasible solution is optimal.

**Franklin Section 1.5, Problem 10** The dual program is
\[
\text{maximize } y^T \begin{bmatrix} 3 + \varepsilon \\
3
\end{bmatrix} \text{ with } y^T \begin{bmatrix} 1 & 1 & 2 \\
2 & 1 & 1
\end{bmatrix} \leq \begin{bmatrix} 0 & 1 & 0
\end{bmatrix}
\]
The equilibrium constraints are that
\[
y^T \begin{bmatrix} 1 \\
2
\end{bmatrix} = 0 \text{ if } x_1 > 0, y^T \begin{bmatrix} 1 \\
1
\end{bmatrix} = 0 \text{ if } x_2 > 0, y^T \begin{bmatrix} 2 \\
1
\end{bmatrix} = 0 \text{ if } x_3 > 0
\]
One possible basic feasible solution is to take \( x_2 = 0 \); then

\[
\begin{bmatrix}
    x_1 \\
    x_3
\end{bmatrix} = \begin{bmatrix}
    1 & 2 \\
    2 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
    3 + \varepsilon \\
    3
\end{bmatrix} = -\frac{1}{3} \begin{bmatrix}
    1 & -2 \\
    -2 & 1
\end{bmatrix} \begin{bmatrix}
    3 + \varepsilon \\
    3
\end{bmatrix} = \begin{bmatrix}
    1 - \frac{1}{3}\varepsilon \\
    1 + \frac{2}{3}\varepsilon
\end{bmatrix}
\]

which is feasible for \( \varepsilon \) sufficiently small (in the interval \([-\frac{3}{2}, 3]\)). This is optimal since then the equilibrium constraints are satisfied by the feasible solution \( y = 0 \). Alternatively, a Phase II calculation could be made involving the second column of the matrix, which would lead to the same result. In the limit as \( \varepsilon \to 0 \), the optimal solution goes to \( x = \begin{bmatrix}
    1 \\
    0 \\
    1
\end{bmatrix} \).

**Franklin Section 1.6, Problem 5** The closed disc is distance 4 units from the origin, so \( \delta = 4 \). The closest point to the origin, \( x_0 \), is \((-\frac{12}{5}, \frac{16}{5})\). The unit vector in the direction of the closest point, \( u \), is \((-\frac{3}{5}, \frac{4}{5})\). The midpoint of the line segment from the origin to \( x_0 \), \( m \), is \((-\frac{6}{5}, \frac{8}{5})\). The equation of a separating plane is then \( u^T(x - m) = 0 \), or

\[
\begin{bmatrix}
    -\frac{3}{5} & \frac{4}{5}
\end{bmatrix} \begin{bmatrix}
    x - \begin{bmatrix}
        -\frac{6}{5} \\
        \frac{8}{5}
    \end{bmatrix}
\end{bmatrix} = 0
\]