Orthogonal Polynomials: Gram-Schmidt process

Thm: The set of polynomial functions $\{\phi_0, \cdots, \phi_n\}$ defined below on $[a, b]$ is orthogonal with respect to the weight function w.

$$\phi_0(x) = 1, \quad \phi_1(x) = x - B_1, \quad \text{and for } k \geq 2$$

$$\phi_k(x) = (x - B_k)\, \phi_{k-1}(x) - C_k\, \phi_{k-2}(x), \quad \text{with}$$

$$B_j = \frac{\int_a^b x\, w(x)\, \phi_{j-1}^2(x)\, dx}{\int_a^b w(x)\, \phi_{j-1}^2(x)\, dx}, \quad j = 1, 2, \cdots, n,$$

$$C_j = \frac{\int_a^b x\, w(x)\, \phi_{j-1}(x)\, \phi_{j-2}(x)\, dx}{\int_a^b w(x)\, \phi_{j-2}^2(x)\, dx}, \quad j = 2, 3, \cdots, n.$$

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Chebyshev Polynomials: $w(x) = \dfrac{1}{\sqrt{1 - x^2}}$ on $[-1, 1]$

§8.3 Chebyshev Polynomials/Power Series Economization

Chebyshev: Gram-Schmidt for orthogonal polynomial functions $\{\phi_0, \cdots, \phi_n\}$ on $[-1, 1]$ with weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

- $\phi_0(x) = 1$; $\phi_1(x) = x - B_1$, with $B_1 = \frac{\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} \, dx}{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx} = 0$.

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- Let $x = \cos(\theta)$, $\theta \in [0, \pi]$. Then $\phi_j(x) = \cos(j\theta)$, $j = 0, 1.$
- Induction hypothesis with $x = \cos(\theta)$, $\theta \in [0, \pi]$:
  $\phi_j(x) = \frac{\cos(j\theta)}{2^{j-1}}$ for $j = 2, \cdots, n-1$.

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  $\phi_j(x) = \frac{\mathbf{cos}(j\theta)}{2^{j-1}}$ for $j = 2, \cdots, n-1$.
- By Gram-Schmidt, for $n \geq 2$

$$\phi_n(x) = (x - B_n)\, \phi_{n-1}(x) - C_n\, \phi_{n-2}(x), \quad \text{with}$$

$$B_n = \frac{\int_{-1}^{1} x\, w(x)\, \phi_{n-1}^2(x)\, dx}{\int_{-1}^{1} w(x)\, \phi_{n-1}^2(x)\, dx},$$

$$C_n = \frac{\int_{-1}^{1} x\, w(x)\, \phi_{n-1}(x)\, \phi_{n-2}(x)\, dx}{\int_{-1}^{1} w(x)\, \phi_{n-2}^2(x)}.$$

Chebyshev Polynomials: Compute B_n

Let $x = \cos(\theta)$, $\theta \in [0, \pi]$. Then $\phi_j(x) = \frac{\cos(j\theta)}{2^{j-1}}$,

$$d\,x = -\sin(\theta)\ d\,\theta, \quad w(x) = \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sin(\theta)}.$$

$$\begin{aligned} B_n &= \frac{\int_0^\pi \frac{\cos(\theta)\cos^2((n-1)\theta)}{\sin(\theta)} \sin(\theta)\ d\,\theta}{\int_0^\pi \frac{\cos^2((n-1)\theta)}{\sin(\theta)} \sin(\theta)\ d\,\theta} \\ &= \frac{\int_0^\pi \cos(\theta)\cos^2((n-1)\theta)\ d\,\theta}{\int_0^\pi \cos^2((n-1)\theta)\ d\,\theta}, \quad \text{where} \end{aligned}$$

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$$\cos(\theta)\cos^2((n-1)\theta) = \frac{1}{2}\cos(\theta)\left(1 + \cos(2(n-1)\theta)\right)$$

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Chebyshev Polynomials: Compute C_n (I)

Let $x = \cos(\theta)$, $\theta \in [0, \pi]$. Then $\phi_j(x) = \frac{\cos(j\theta)}{2^{j-1}}$,

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$$\begin{aligned} C_n &= \frac{\int_0^\pi \frac{\cos(\theta)\cos((n-1)\theta)\cos((n-2)\theta)}{\sin(\theta)} \sin(\theta)\ d\,\theta}{2\ \int_0^\pi \frac{\cos^2((n-2)\theta)}{\sin(\theta)} \sin(\theta)\ d\,\theta} \\ &= \frac{\int_0^\pi \cos(\theta)\cos((n-1)\theta)\ \cos((n-2)\theta)\ d\,\theta}{2\ \int_0^\pi \cos^2((n-2)\theta)\ d\,\theta}, \quad \text{where} \end{aligned}$$

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$$\begin{aligned} &\cos(\theta)\cos((n-1)\theta)\cos((n-2)\theta) \\ &= \frac{1}{2}\cos((n-1)\theta)\left(\cos((n-1)\theta) + \cos((n-3)\theta)\right) \\ &= \frac{1}{4}\left(1 + \cos(2(n-1)\theta) + \cos(2(n-2)\theta) + \cos(2\theta)\right), \quad \text{so} \end{aligned}$$

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$$\int_0^\pi \cos(\theta)\cos((n-1)\theta)\cos((n-2)\theta)\ d\,\theta = \begin{cases} \frac{\pi}{2}, & \text{for } n = 2, \\ \frac{\pi}{4}, & \text{for } n > 2. \end{cases}$$

Chebyshev Polynomials: Compute C_n (II)

$$C_n = \frac{\int_0^\pi \cos(\theta)\cos((n-1)\theta)\ \cos((n-2)\theta)\ d\,\theta}{2\ \int_0^\pi \cos^2((n-2)\theta)\ d\,\theta}, \quad \text{where}$$

Chebyshev Polynomials: Compute C_n (II)

$$C_n = \frac{\int_0^{\pi} \mathbf{cos}\,(\theta)\,\mathbf{cos}\,((n-1)\theta)\ \mathbf{cos}\,((n-2)\theta)\ d\,\theta}{2\ \int_0^{\pi} \mathbf{cos}^2\,((n-2)\theta)\ d\,\theta}, \quad \text{where}$$

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- Induction on $\phi_n(x)$: With $x = \cos(\theta)$,

$$\begin{aligned} \phi_n(x) &= \frac{\cos(\theta)\cos((n-1)\theta)}{2^{n-2}} - \frac{1}{4}\frac{\cos((n-2)\theta)}{2^{n-3}} \\ &= \frac{\cos(n\theta)+\cos((n-2)\theta)}{2^{n-1}} - \frac{\cos((n-2)\theta)}{2^{n-1}} \\ &= \frac{\cos(n\theta)}{2^{n-1}}. \end{aligned}$$

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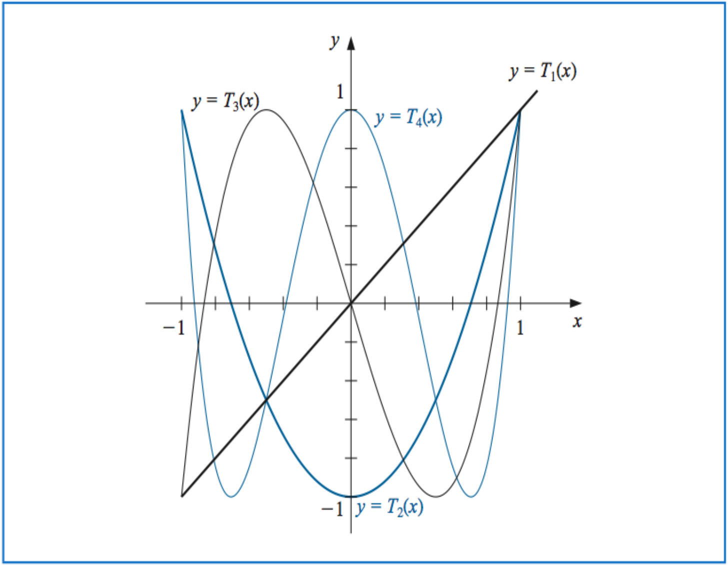
- Chebyshev Polynomials: $T_0(x) = 1$, $T_1(x) = \cos(\theta)$, $T_n(x) = \cos(n\theta)$, $n = 2, 3, \cdots$.

Chebyshev Polynomials: $T_1(x)$ through $T_4(x)$



Chebyshev Polynomials: Zeros and extrema in $[-1, 1]$

Let $x = \cos(\theta)$, $\theta \in [0, \pi]$, $T_0(x) = 1$, and for $n \geq 1$,

$$T_n(x) = \cos(n\theta) = 2^{n-1} x^n + \text{lower order terms.}$$

- All zeros are simple and in $(-1, 1)$: $T_n\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right) = 0$ for $k = 1, \cdots, n$.
- $n + 1$ local extrema on $[-1, 1]$: $T_n\left(\cos\left(\frac{k}{n}\pi\right)\right) = (-1)^k$ for $k = 0, 1, \cdots, n$.
- $T_n(x)$ is monotonic for $|x| \geq 1$,

$$T_n(x) = \frac{1}{2}\left(\left(x - \sqrt{x^2 - 1}\right)^n + \left(x + \sqrt{x^2 - 1}\right)^n\right).$$

Chebyshev Polynomials: Min-Max Theorem

Let $\widetilde{\Pi}_n$ denote the set of all MONIC polynomials of degree n. Then

$$\mathbf{min}_{P_n(x)\in\widetilde{\Pi}_n}\left(\mathbf{max}_{x\in[-1,1]}\left|P_n(x)\right|\right)=\mathbf{max}_{x\in[-1,1]}\left|\frac{T_n(x)}{2^{n-1}}\right|=\frac{1}{2^{n-1}}.$$

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Proof: First we have $\dfrac{T_n(x)}{2^{n-1}} \in \widetilde{\Pi}_n$ with $\dfrac{T_n\left(\mathbf{cos}\left(\frac{k}{n}\pi\right)\right)}{2^{n-1}} = \dfrac{(-1)^k}{2^{n-1}}$.

Therefore $\quad \mathbf{min}_{P_n(x)\in\widetilde{\Pi}_n}\left(\mathbf{max}_{x\in[-1,1]}\left|P_n(x)\right|\right) \leq \dfrac{1}{2^{n-1}}$.

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Therefore $\quad \min_{P_n(x)\in\widetilde{\Pi}_n}\left(\max_{x\in[-1,1]}|P_n(x)|\right)\leq\dfrac{1}{2^{n-1}}$.

Let $P_n(x)\in\widetilde{\Pi}_n$ be a polynomial with $\max_{x\in[-1,1]}|P_n(x)|\leq\frac{1}{2^{n-1}}$.

Then $Q(x)\stackrel{def}{=}\dfrac{T_n(x)}{2^{n-1}}-P_n(x)$ has degree at most $n-1$, satisfies

$$(-1)^k\;Q\left(\cos\left(\frac{k}{n}\pi\right)\right)=\frac{1}{2^{n-1}}-(-1)^k\;P_n\left(\cos\left(\frac{k}{n}\pi\right)\right)\geq 0.$$

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$$(-1)^k\ Q\left(\cos\left(\frac{k}{n}\pi\right)\right) = \frac{1}{2^{n-1}} - (-1)^k\ P_n\left(\cos\left(\frac{k}{n}\pi\right)\right) \ge 0.$$

Since $0 \le k \le n$, $Q(x)$ changes sign or reaches 0 at least n times, this can be true only if $Q(x) \equiv 0$, or $P_n(x) = \frac{T_n(x)}{2^{n-1}}$. Thus

$\max_{x\in[-1,1]}|P_n(x)| = \dfrac{1}{2^{n-1}}$. $\square$

Reducing the Degree of Approximating Polynomials

Given an arbitrary n^{th}-degree polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0,$$

a (somewhat silly) problem is to find a polynomial $P_{n-1}(x)$ of degree at most $(n-1)$ to best approximate $P_n(x)$ on $[-1, 1]$:

$$P_{n-1}(x) = \mathbf{argmin}_{P(x) \in \Pi_{n-1}} \left(\mathbf{max}_{x \in [-1,1]} \left| P_n(x) - P(x) \right| \right).$$

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Since the set of polynomials of the form $(P_n(x) - P(x))/a_n$ is precisely $\widetilde{\Pi}_n$, it follows from Min-Max Theorem that

$$(P_n(x) - P(x))/a_n = \frac{T_n(x)}{2^{n-1}}$$

gives the best approximation. So the best approximation

$$P_{n-1}(x) = P_n(x) - \frac{a_n T_n(x)}{2^{n-1}}, \quad \text{with} \quad |P_n(x) - P_{n-1}(x)| \leq \frac{|a_n|}{2^{n-1}}.$$

Maclaurin Polynomial vs. Chebyshev Polynomial

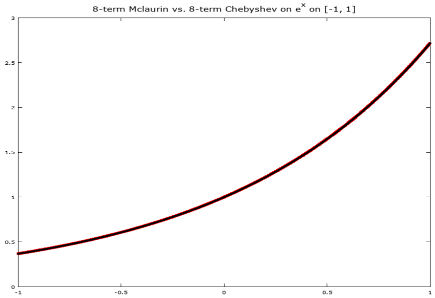
Given function $f(x)$ on $[-1,1]$, the n^{th}-degree Maclaurin polynomial is

$$P_n(x) \stackrel{def}{=} \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j,$$

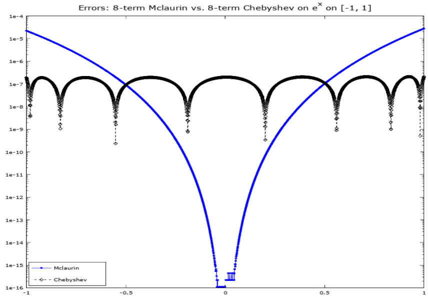
and the n^{th}-degree Chebyshev approximation is

$$\widehat{P}_n(x) \stackrel{def}{=} \sum_{j=0}^{n} \alpha_j T_j(x),$$

$$\text{with} \quad \alpha_j = \frac{\int_{-1}^{1} \frac{f(x)\, T_j(x)}{\sqrt{1-x^2}}\, d\,x}{\int_{-1}^{1} \frac{T_j^2(x)}{\sqrt{1-x^2}}\, d\,x} = \frac{\int_0^{\pi} f(\mathbf{cos}(\theta))\, \mathbf{cos}(j\,\theta)\, d\,\theta}{\int_0^{\pi} \mathbf{cos}^2(j\,\theta)\, d\,\theta}.$$


8-term Mclaurin vs. 8-term Chebyshev on e^x on [-1, 1]
3
2.5
2
1.5
1
0.5
0
-1
-0.5
0
0.5
1

8-term Mclaurin vs. 8-term Chebyshev on e^x on [-1, 1]



Errors: 8-term Mclaurin vs. 8-term Chebyshev on e^x on [-1, 1]
Mclaurin
Chebyshev

§8.4 Rational Function Approximation

A RATIONAL FUNCTION r of degree N has the form

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials whose degrees sum to N.

Pro Rational functions can better approximate a given function than polynomials.

Con Denominator function $q(x)$ may have un-wanted zeros to mess up the over-all approximation.

Padé Approximation (I)

Let r be a rational function of degree $N = n + m$ of the form

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + \cdots + p_n x^n}{q_0 + q_1 x + \cdots + q_m x^m}.$$

PADÉ APPROXIMATION OF GIVEN FUNCTION $f(x)$: Choose coefficients $p_0, p_1, \cdots, p_n$ and $q_0, q_1, \cdots, q_m$ so that

$$f^{(k)}(0) = r^{(k)}(0), \quad k = 0, 1, \cdots, N.$$

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- Must have $q_0 \neq 0$. Could set $q_0 = 1$ or any non-zero value.
- There must exist a continuous function $h(x)$ such that

$$f(x) - r(x) = \frac{f(x)\ q(x) - p(x)}{q(x)} = \frac{x^{N+1} h(x)}{q(x)},$$

i.e., $f - r$ has a zero of multiplicity $N + 1$ at $x = 0$.

Padé Approximation (II)

Assume Maclaurin series expansion $f(x) = \sum_{i=0}^{\infty} a_i\, x^i$. Let $p_j = 0$ for $j \geq n+1$ and $q_k = 0$ for $k \geq m+1$.

$$\begin{aligned} f(x)\; q(x) - p(x) &= \left(\sum_{i=0}^{\infty} a_i\, x^i\right) \left(\sum_{j=0}^{\infty} q_j\, x^j\right) - \left(\sum_{k=0}^{\infty} p_k\, x^k\right) \\ &= \sum_{k=0}^{\infty} \left(\left(\sum_{i+j=k} a_i\, q_j \right) - p_k \right) x^k = x^{N+1}\, h(x). \end{aligned}$$

Padé equations:

$$\begin{aligned} \sum_{j=0}^{\min(k,m)} a_{k-j}\, q_j &= p_k, \quad k = 0, 1, \cdots n, \\ \sum_{j=0}^{\min(k,m)} a_{k-j}\, q_j &= 0, \quad k = n+1, \cdots N. \end{aligned}$$

Chebyshev Rational Function Approximation (I)

Assume Chebyshev polynomial expansion $f(x) = \sum_{i=0}^{\infty} a_i \, T_i(x)$.
Choose Chebyshev polynomial expressions for $p(x)$ and $q(x)$

$$p(x) = \sum_{k=0}^{n} p_k \, T_k(x) \stackrel{def}{=} \sum_{k=0}^{\infty} p_k \, T_k(x),$$

$$q(x) = \sum_{j=0}^{m} q_j \, T_j(x) \stackrel{def}{=} \sum_{j=0}^{\infty} q_j \, T_j(x), \quad \text{so that}$$

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$$q(x) = \sum_{j=0}^{m} q_j \, T_j(x) \stackrel{def}{=} \sum_{j=0}^{\infty} q_j \, T_j(x), \quad \text{so that}$$

$$\begin{aligned} f(x) - r(x) &= \frac{f(x)\, q(x) - p(x)}{q(x)} \\ &= \frac{\left(\sum_{i=0}^{\infty} a_i \, T_i(x)\right)\left(\sum_{j=0}^{m} q_j \, T_j(x)\right) - \left(\sum_{k=0}^{n} p_k \, T_k(x)\right)}{q(x)} \\ &= \frac{\left(\sum_{i,j=0}^{\infty} a_i \, q_j \, T_i(x)\, T_j(x)\right) - \left(\sum_{k=0}^{n} p_k \, T_k(x)\right)}{q(x)}, \end{aligned}$$

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$$f(x) - r(x) = \frac{f(x)\; q(x) - p(x)}{q(x)}$$

$$= \frac{\left(\sum_{i=0}^{\infty} a_i \, T_i(x)\right)\left(\sum_{j=0}^{m} q_j \, T_j(x)\right) - \left(\sum_{k=0}^{n} p_k \, T_k(x)\right)}{q(x)}$$

$$= \frac{\left(\sum_{i,j=0}^{\infty} a_i \, q_j \, T_i(x)\; T_j(x)\right) - \left(\sum_{k=0}^{n} p_k \, T_k(x)\right)}{q(x)},$$

$$\underline{\underline{\textbf{Goal}}} \quad \frac{\sum_{k=N+1}^{\infty} h_k \, T_k(x)}{q(x)}.$$

Chebyshev Rational Function Approximation (II)

Chebyshev polynomial identity: $T_i(x)\;T_j(x) = \frac{1}{2}\left(T_{i+j}(x) + T_{|i-j|}(x)\right)$.

$$\sum_{i,j=0}^{\infty} a_i\, q_j\, T_i(x)\;T_j(x) = \left(a_0\, q_0 + \frac{1}{2}\sum_{j=1}^{m} a_j\, q_j\right) T_0(x)$$

$$+\frac{1}{2}\sum_{k=1}^{\infty}\left(\left(\sum_{i+j=k} a_i\, q_j\right) + \left(\sum_{j=0}^{\infty} a_{j+k}\, q_j\right) + \left(\sum_{i=0}^{\infty} a_i\, q_{i+k}\right)\right)\; T_k(x)\,.$$

Chebyshev Rational Function Approximation (II)

Chebyshev polynomial identity: $T_i(x)\ T_j(x) = \frac{1}{2}\left(T_{i+j}(x) + T_{|i-j|}(x)\right).$

$$\sum_{i,j=0}^{\infty} a_i\, q_j\, T_i(x)\ T_j(x) = \left(a_0\, q_0 + \frac{1}{2}\sum_{j=1}^{m} a_j\, q_j\right) T_0(x)$$

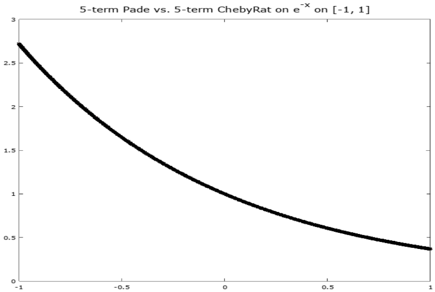
$$+\frac{1}{2}\sum_{k=1}^{\infty}\left(\left(\sum_{i+j=k} a_i\, q_j\right) + \left(\sum_{j=0}^{\infty} a_{j+k}\, q_j\right) + \left(\sum_{i=0}^{\infty} a_i\, q_{i+k}\right)\right) T_k(x)\,.$$

Chebyshev rational function equations:

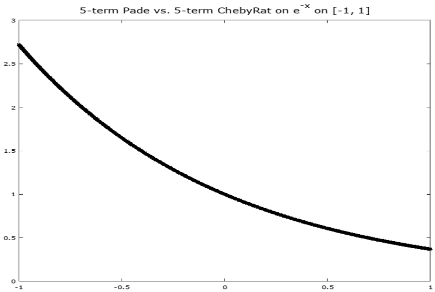
$$2\, p_0 = 2\, a_0\, q_0 + \sum_{j=1}^{m} a_j\, q_j,$$

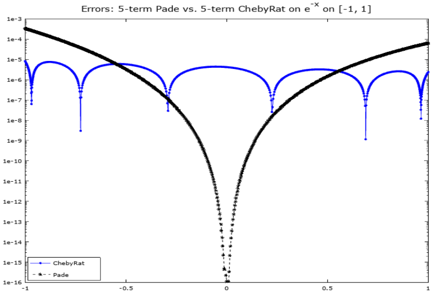
$$2\, p_k = \sum_{j=0}^{\mathbf{min}(k,m)} a_{k-j}\, q_j + \sum_{j=0}^{m} a_{j+k}\, q_j + \sum_{i=0}^{m-k} a_i\, q_{i+k},\ 1 \le k \le n,$$

$$0 = \sum_{j=0}^{\mathbf{min}(k,m)} a_{k-j}\, q_j + \sum_{j=0}^{m} a_{j+k}\, q_j + \sum_{i=0}^{m-k} a_i\, q_{i+k},\ n+1 \le k \le N.$$



5-term Pade vs. 5-term ChebyRat on e^-x on [-1, 1]
3
2.5
2
1.5
1
0.5
0
-1
-0.5
0
0.5
1


5-term Pade vs. 5-term ChebyRat on e^-x on [-1, 1]


Errors: 5-term Pade vs. 5-term ChebyRat on e^-x on [-1, 1]
ChebyRat
Pade

§8.5 Trigonometric Polynomial Approximation

- TRIGONOMETRIC POLYNOMIALS of degree $\leq n$:

$$\begin{aligned}\phi_0(x) &= \frac{1}{2}, \quad \phi_k(x) = \cos(k\,x), \quad \text{for} \quad k = 1, \cdots, n, \\ \phi_{n+k}(x) &= \sin(k\,x), \quad \text{for} \quad k = 1, \cdots, n.\end{aligned}$$

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$$\phi_{n+k}(x) = \sin(k\,x), \quad \text{for} \quad k = 1, \cdots, n.$$

- $\{\phi_k(x)\}_{k=0}^{2n}$ orthogonal on $[-\pi, \pi]$ for weight function $w(x) \equiv 1$.

$$\int_{-\pi}^{\pi} \phi_k(x)\ \phi_j(x)\ d\,x = 0 \quad \text{for all } k \ne j. \text{ i.e.,}$$

$$\int_{-\pi}^{\pi} \frac{1}{2}\cos(k\,x)\ d\,x = \int_{-\pi}^{\pi} \frac{1}{2}\sin(k\,x)\ d\,x = 0,$$

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$$\int_{-\pi}^{\pi} \cos(k\,x)\,\cos(j\,x)\,d\,x = \frac{1}{2}\left(\int_{-\pi}^{\pi} (\cos((k+j)\,x) + \cos((k-j)\,x))\,d\,x\right) = 0,$$

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$$\int_{-\pi}^{\pi} \sin(k\,x)\sin(j\,x)\,d\,x = \frac{1}{2}\left(\int_{-\pi}^{\pi}(\cos((k-j)\,x) - \cos((k+j)\,x))\,d\,x\right) = 0,$$

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$$\int_{-\pi}^{\pi} \frac{1}{2} \cos(k\,x)\, d\,x = \int_{-\pi}^{\pi} \frac{1}{2} \sin(k\,x)\, d\,x = 0,$$
$$\int_{-\pi}^{\pi} \cos(k\,x)\, \cos(j\,x)\, d\,x = \frac{1}{2}\left(\int_{-\pi}^{\pi} (\cos((k+j)\,x) + \cos((k-j)\,x))\, d\,x\right) = 0,$$
$$\int_{-\pi}^{\pi} \sin(k\,x)\, \sin(j\,x)\, d\,x = \frac{1}{2}\left(\int_{-\pi}^{\pi} (\cos((k-j)\,x) - \cos((k+j)\,x))\, d\,x\right) = 0,$$
$$\int_{-\pi}^{\pi} \sin(k\,x)\, \cos(j\,x)\, d\,x = \frac{1}{2}\left(\int_{-\pi}^{\pi} (\sin((k+j)\,x) + \sin((k-j)\,x))\, d\,x\right) = 0.$$

- Orthogonal trigonometric polynomials:

$$\begin{aligned} \phi_0(x) &= \frac{1}{2}, \quad \phi_k(x) = \cos(x), \quad \text{for} \quad k = 1, \cdots, n, \\ \phi_{n+k}(x) &= \sin(x), \quad \text{for} \quad k = 1, \cdots, n. \end{aligned}$$

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$$\phi_{n+k}(x) = \mathbf{sin}(x), \quad \text{for} \quad k = 1, \cdots, n.$$

- Given function $f \in C[-\pi, \pi]$, continuous least squares approximation by trigonometric polynomials:

$$\mathbf{min}_{S_n(x)} \int_{-\pi}^{\pi} \left(f(x) - S_n(x)\right)^2 \, d\,x, \quad \text{where}$$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \, \mathbf{cos}(k\,x) + b_k \mathbf{sin}(k\,x)\right).$$

$$\int_{-\pi}^{\pi} (f(x) - S_n(x))^2 \, dx$$

$$= \int_{-\pi}^{\pi} \left(f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \right) \right)^2 \, dx$$

$$\overset{def}{=} \Delta + \int_{-\pi}^{\pi} f^2(x) \, dx - 2 \int_{-\pi}^{\pi} f(x) \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \right) dx,$$

where $\Delta = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \right)^2 \, dx$

$$= \left(\frac{a_0}{2} \right)^2 \int_{-\pi}^{\pi} dx + \sum_{k=1}^{n} \left(a_k^2 \int_{-\pi}^{\pi} \cos^2(kx) \, dx + b_k^2 \int_{-\pi}^{\pi} \sin^2(kx) \, dx \right)$$

$$= \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right).$$

$$\int_{-\pi}^{\pi} (f(x) - S_n(x))^2 \, dx = \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right)$$

$$+ \int_{-\pi}^{\pi} f^2(x) \, dx - 2 \int_{-\pi}^{\pi} f(x) \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \right) dx,$$

Optimal solution to $\mathbf{min}_{S_n(x)} \int_{-\pi}^{\pi} (f(x) - S_n(x))^2 \, dx$:

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, dx, \quad \text{for} \quad k = 0, 1, \cdots, n, \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx, \quad \text{for} \quad k = 1, \cdots, n. \end{aligned}$$

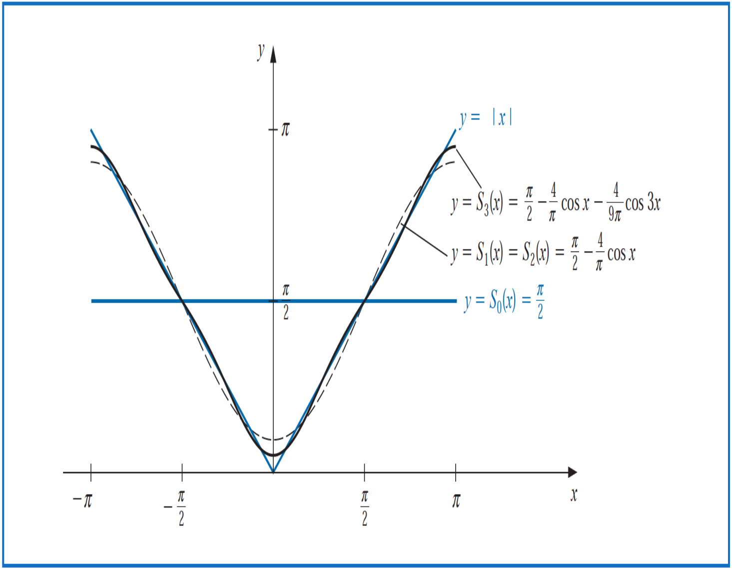
Ex: Approximating $f(x) = |x|$ on $[-\pi, \pi]$ (I)

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \, dx = \frac{2}{\pi}\int_{0}^{\pi} x \, dx = \pi, \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(k\,x) \, dx = \frac{2}{\pi}\int_{0}^{\pi} x \cos(k\,x) \, dx \\
&= \frac{2}{\pi\,k^2}\left((-1)^k - 1\right), \quad \text{for} \quad k = 1, \cdots, n, \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin(k\,x) \, dx = 0, \quad \text{for} \quad k = 1, \cdots, n.
\end{aligned}
$$

Trigonometric approximation of degree n

$$
S_n(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(k\,x).
$$

Ex: Approximating $f(x) = |x|$ on $[-\pi, \pi]$ (II)



Discrete Trigonometric Least Squares

Given: Specific data pairs $\{(x_j, y_j)\}_{j=0}^{2m-1}$, with $x_j = \dfrac{j-m}{m}\pi$ for $j = 0, 1, \cdots, 2m-1$. With basis functions

$$\begin{aligned} \phi_0(x) &= 1, \quad \phi_k(x) = \cos(x), \quad \text{for} \quad k = 1, \cdots, n, \\ \phi_{n+k}(x) &= \sin(x), \quad \text{for} \quad k = 1, \cdots, n. \end{aligned}$$

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Goal: to determine best trigonometric polynomial

$$\min_{\alpha_0, \alpha_1, \cdots, \alpha_{2m}} \sum_{j=0}^{2\,m} \left(y_j - \left(\frac{\alpha_0}{2} \phi_0(x_j) + \sum_{k=1}^{2\,m} \alpha_k \phi_k(x_j) \right) \right)^2.$$

$$\sum_{j=0}^{2\,m} \left(y_j - \left(\frac{\alpha_0}{2} \phi_0(x_j) + \sum_{k=1}^{2\,m} \alpha_j \phi_k(x_j) \right) \right)^2$$

$$\begin{aligned} = \quad & \left(\sum_{j=0}^{2\,m} y_j^2 \right) - 2 \left(\sum_{j=0}^{2\,m} y_j \left(\frac{\alpha_0}{2} \phi_0(x_j) + \sum_{k=1}^{2\,m} \alpha_k \phi_k(x_j) \right) \right) \\ & + \sum_{j=0}^{2\,m} \left(\frac{\alpha_0}{2} \phi_0(x_j) + \sum_{k=1}^{2\,m} \alpha_k \phi_k(x_j) \right)^2 \end{aligned}$$

Discrete Orthogonality: $\sum_{j=0}^{2m-1} \phi_k(x_j)\, \phi_r(x_j) = 0$ for $k \neq r$

Discrete data points $x_j = \frac{j-m}{m}\pi$ for $j = 0, 1, \cdots, 2m-1$.

$$
\begin{aligned}
\sum_{j=0}^{2m-1} \cos(r\,x_j) &= 0, \quad r = 1, \cdots, m, \\
\sum_{j=0}^{2m-1} \sin(r\,x_j) &= 0, \quad r = 1, \cdots, m, \\
\sum_{j=0}^{2m-1} \cos(r\,x_j)\cos(k\,x_j) &= 0, \quad r \neq k,\ r, k = 1, \cdots, m, \\
\sum_{j=0}^{2m-1} \sin(r\,x_j)\sin(k\,x_j) &= 0, \quad r \neq k,\ r, k = 1, \cdots, m, \\
\sum_{j=0}^{2m-1} \cos(r\,x_j)\sin(k\,x_j) &= 0, \quad r, k = 1, \cdots, m.
\end{aligned}
$$

Proof of Orthogonality: $x_j = \dfrac{j - m}{m}\,\pi$ for $j = 0, 1, \cdots, 2\,m - 1$.

$$\left(\sum_{j=0}^{2\,m-1} \mathbf{cos}\,(r\,x_j)\right) + \sqrt{-1}\left(\sum_{j=0}^{2\,m-1} \mathbf{sin}\,(r\,x_j)\right)$$

Proof of Orthogonality: $x_j = \frac{j-m}{m}\pi$ for $j = 0, 1, \cdots, 2m-1$.

$$\left(\sum_{j=0}^{2m-1} \cos(r\,x_j)\right) + \sqrt{-1}\left(\sum_{j=0}^{2m-1} \sin(r\,x_j)\right)$$

$$= \sum_{j=0}^{2m-1} \exp\left(\sqrt{-1}\,r\,x_j\right) = \exp\left(-\sqrt{-1}\,r\,\pi\right)\sum_{j=0}^{2m-1} \exp\left(\sqrt{-1}\,\frac{r\,\pi}{m}\,j\right)$$

$$= \exp\left(-\sqrt{-1}\,r\,\pi\right)\frac{1-\exp\left(\sqrt{-1}\,\frac{r\,\pi}{m}\,2\,m\right)}{1-\exp\left(\sqrt{-1}\,\frac{r\,\pi}{m}\right)} = 0, \quad \text{therefore}$$

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$$\left(\sum_{j=0}^{2m-1} \cos(r\,x_j)\right) + \sqrt{-1}\left(\sum_{j=0}^{2m-1} \sin(r\,x_j)\right)$$

$$= \sum_{j=0}^{2m-1} \exp\left(\sqrt{-1}\,r\,x_j\right) = \exp\left(-\sqrt{-1}\,r\,\pi\right) \sum_{j=0}^{2m-1} \exp\left(\sqrt{-1}\,\frac{r\,\pi}{m}\,j\right)$$

$$= \exp\left(-\sqrt{-1}\,r\,\pi\right) \frac{1-\exp\left(\sqrt{-1}\,\frac{r\,\pi}{m}\,2\,m\right)}{1-\exp\left(\sqrt{-1}\,\frac{r\,\pi}{m}\right)} = 0, \quad \text{therefore}$$

$$\sum_{j=0}^{2m-1} \cos(r\,x_j) = \sum_{j=0}^{2m-1} \sin(r\,x_j) = 0.$$

Proof of Orthogonality: $x_j = \frac{j - m}{m}\pi$ for $j = 0, 1, \cdots, 2m - 1$.

$$\left(\sum_{j=0}^{2m-1} \cos(r x_j)\right) + \sqrt{-1}\left(\sum_{j=0}^{2m-1} \sin(r x_j)\right)$$

$$= \sum_{j=0}^{2m-1} \exp\left(\sqrt{-1}\, r\, x_j\right) = \exp\left(-\sqrt{-1}\, r\, \pi\right) \sum_{j=0}^{2m-1} \exp\left(\sqrt{-1}\, \frac{r\pi}{m}\, j\right)$$

$$= \exp\left(-\sqrt{-1}\, r\, \pi\right) \frac{1 - \exp\left(\sqrt{-1}\, \frac{r\pi}{m}\, 2m\right)}{1 - \exp\left(\sqrt{-1}\, \frac{r\pi}{m}\right)} = 0, \quad \text{therefore}$$

$$\sum_{j=0}^{2m-1} \cos(r x_j) = \sum_{j=0}^{2m-1} \sin(r x_j) = 0.$$

$$\sum_{j=0}^{2m-1} \cos(r x_j) \cos(k x_j)$$

$$= \frac{1}{2}\left(\left(\sum_{j=0}^{2m-1} \cos((r + k) x_j)\right) + \left(\sum_{j=0}^{2m-1} \cos((r - k) x_j)\right)\right) = 0.$$

Discrete Trigonometric Least Squares

$$\sum_{j=0}^{2m-1}\left(y_j-\left(\frac{\alpha_0}{2}\,\phi_0\left(x_j\right)+\sum_{k=1}^{2m}\alpha_k\,\phi_k\left(x_j\right)\right)\right)^2$$

$$=\quad\left(\sum_{j=0}^{2m-1}y_j^2\right)-2\left(\sum_{j=0}^{2m-1}y_j\left(\frac{\alpha_0}{2}\,\phi_0\left(x_j\right)+\sum_{k=1}^{2m}\alpha_k\,\phi_k\left(x_j\right)\right)\right)+\Delta,$$

$$\text{where}\quad\Delta=\sum_{j=0}^{2m-1}\left(\frac{\alpha_0}{2}\,\phi_0\left(x_j\right)+\sum_{k=1}^{2m}\alpha_k\,\phi_k\left(x_j\right)\right)^2$$

$$=\left(\frac{\alpha_0}{2}\right)^2\sum_{j=0}^{2m-1}\phi_0^2\left(x_j\right)+\sum_{k=1}^{2m}\alpha_k^2\left(\sum_{j=0}^{2m-1}\phi_k\left(x_j\right)^2\right)$$

$$=m\left(\frac{\alpha_0^2}{2}+\sum_{k=1}^{2m}\alpha_k^2\right)$$

Discrete Trigonometric Least Squares

$$\sum_{j=0}^{2m-1}\left(y_j-\left(\frac{\alpha_0}{2}\,\phi_0(x_j)+\sum_{k=1}^{2m}\alpha_k\,\phi_k(x_j)\right)\right)^2$$

$$=\left(\sum_{j=0}^{2m-1}y_j^2\right)-2\left(\sum_{j=0}^{2m-1}y_j\left(\frac{\alpha_0}{2}\,\phi_0(x_j)+\sum_{k=1}^{2m}\alpha_k\,\phi_k(x_j)\right)\right)$$

$$+m\left(\frac{\alpha_0^2}{2}+\sum_{k=1}^{2m}\alpha_k^2\right).$$

Minimize least squares error,

$$\alpha_k=\frac{1}{m}\sum_{j=0}^{2m-1}y_j\,\phi_k(x_j),\quad k=0,1,\cdots,2m,\quad \text{With}$$

$$\phi_0(x)=1,\quad \phi_k(x)=\cos(x),\quad \text{for}\quad k=1,\cdots,m,$$
$$\phi_{m+k}(x)=\sin(x),\quad \text{for}\quad k=1,\cdots,m.$$

FFT History

- First FFT algorithm by Gauss, 1805

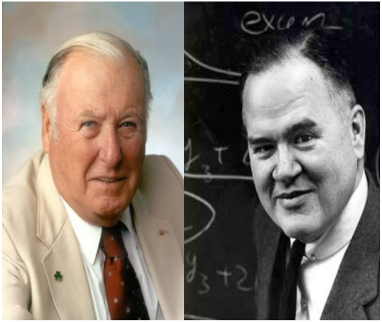
FFT History

- First FFT algorithm by Gauss, 1805

- Fourier analysis by Fourier, 1807

FFT History

- Cooley and Tukey FFT algorithm, 1965



James William Cooley (1926-) John Wilder Tukey (1915-2000)