Self Introduction

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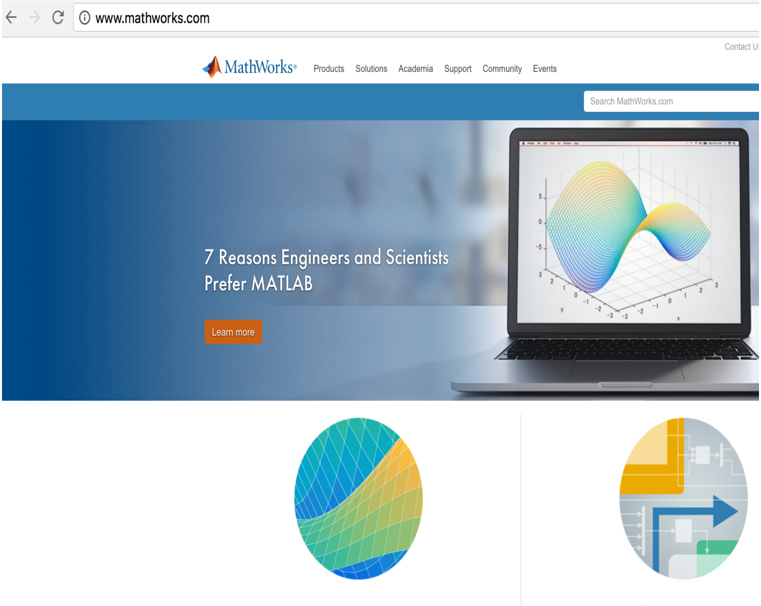
- **Office Hours:** MWF 3:30-5:00PM
- **Class Website:** math.berkeley.edu/∼mgu/MA128BSpring2018

Text Book

- Burden and Faires, **Numerical Analysis**.
  Required. Homework based on 9^{th} edition.

Matlab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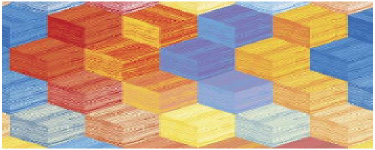
and maybe python

Math 98: Introduction to Matlab

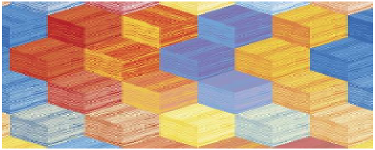
- runs 6 weeks, starting next week;



2018 Spring Class #: 26618

MATH 98BC 001
001 DIS
1 Units
Tu
6:00 pm - 6:59 pm

0 Total Open Seats



2018 Spring Class #: 26619

MATH 98BC 002
002 DIS
1 Units
W
6:00 pm - 6:59 pm

0 Total Open Seats

Material to be covered in class

- First 9 weeks: Chapters 7 through 10 of Text Book, except Section 10.5.
- Remaining 5 weeks: special topics in (randomized) numerical linear algebra. **Paper links on class website.**
- **NO** differential equations.

Class Work

- First 9 weeks: weekly home work sets;
  Count best 8, total 24 points.
- 4 Quizzes;
  Count best 3, total 12 points.
- 1 Project, total 24 points;
- 1 Midterm exam, 20 points;
- 1 Final exam, 20 points.
- FINAL WORTH 40 POINTS IF MIDTERM MISSING.

Quiz and Exam Schedule

- **Quiz**: Jan. 25 in discussion
- **Quiz**: Feb. 8 in discussion
- **Quiz**: Feb. 22 in discussion
- **Quiz**: Mar. 8 in discussion
- **Midterm**: Mar. 22 in class
- **Project Presentation**: Apr. 30, May 2, May 4 in class
- **Final Exam**: Tues., 5/08/18, 11:30–2:30pm (Exam Group 6)

Grade Scale

- **A**- to **A+**: at least 85 points;
- **B**- to **B+**: between 70 and 85 points;
- **C**- to **C+**: between 60 and 70 points;
- **D**: between 55 and 60 points;
- **F**: less than 55 points.

No grade curve; most people get *A* level or *B* level grades.

Gaussian Elimination with partial pivoting (GEPP): Review

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

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- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv}_s \stackrel{def}{=} \mathbf{argmax}_{s \leq j \leq n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}_s}$$

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(PERMUTATION: interchange rows s and $\mathbf{piv}_s$).

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  - **eliminating** x_s from E_{s+1} through E_n:

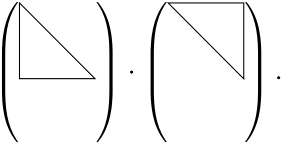
    $$l_{js} \quad = \quad \frac{a_{js}}{a_{ss}}, \quad s+1 \leq j \leq n,$$

    $$a_{jk} \quad \stackrel{\textbf{overwrite}}{=} \quad a_{jk} - l_{js}\, a_{sk}, \quad s+1 \leq j,\ k \leq n.$$

GEPP as LU factorization

Theorem: Let $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ be non-singular. Then GEPP computes an LU factorization with permutation matrix P such that

$$P \cdot A = L \cdot U =$$



GEPP as LU factorization, example

$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \\ 1 & & & \end{pmatrix}.$$

$$P \cdot A = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 & 2 \\ & 1 & 1 & 0 \\ & & 1 & 2 \\ & & & 3 \end{pmatrix} \stackrel{def}{=} L \cdot U.$$

Solving general linear equations with GEPP

$$A\mathbf{x} = \mathbf{b}, \quad P \cdot A = L \cdot U$$

- interchanging components in $\mathbf{b}$

$$P \cdot (A\mathbf{x}) = (P \cdot \mathbf{b}), \quad (L \cdot U)\,\mathbf{x} = (P \cdot \mathbf{b}).$$

- solving for $\mathbf{b}$ with forward and backward substitution

$$\begin{aligned} \mathbf{x} &= (L \cdot U)^{-1} (P \cdot \mathbf{b}) \\ &= \left(U^{-1} \left(L^{-1} (P \cdot \mathbf{b})\right)\right). \end{aligned}$$

Solving general linear equations with GEPP

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Cost Analysis

- computing $P \cdot A = L \cdot U$: about $2/3n^3$ operations.
- forward and backward substitution: about $2n^2$ operations.
- maybe too expensive for large n.

§7.1 Vector Norm

A VECTOR NORM on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

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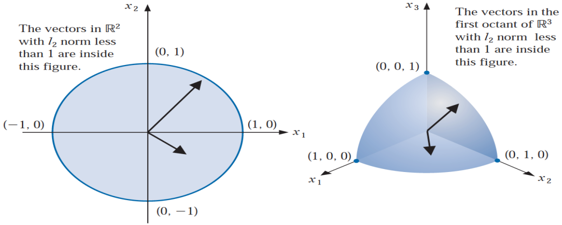
Examples: 2−norm and ∞−norm for $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$:

$$\|\mathbf{x}\|_2 \overset{def}{=} \sqrt{x_1^2 + \cdots + x_n^2}, \quad \text{and} \quad \|\mathbf{x}\|_\infty \overset{def}{=} \mathbf{max}_{1\leq j\leq n}|x_j|.$$

Unit 2−norm and Unit ∞−norm

- Unit 2−norm: unit disk in $\mathbb{R}^2$, unit ball in $\mathbb{R}^3$



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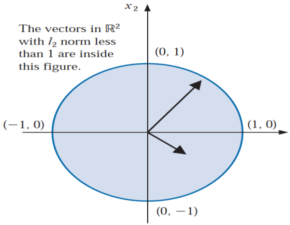
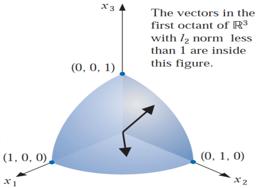
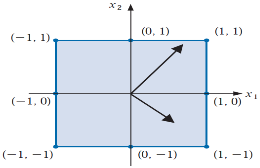
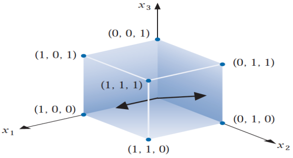




- Unit ∞−norm: unit square in $\mathbb{R}^2$, unit box in $\mathbb{R}^3$:



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.



The vectors in the first octant of $\mathbb{R}^3$ with l_∞ norm less than 1 are inside this figure.

Example: 2−norm and ∞−norm for $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Solution:

$$\begin{aligned} \|\mathbf{x}\|_2 &= \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} = 3.7\cdots, \\ \|\mathbf{x}\|_\infty &= \mathbf{max}\,(1, |-2|, 3) = 3. \end{aligned}$$

Equivalence of 2−norm and ∞−norm

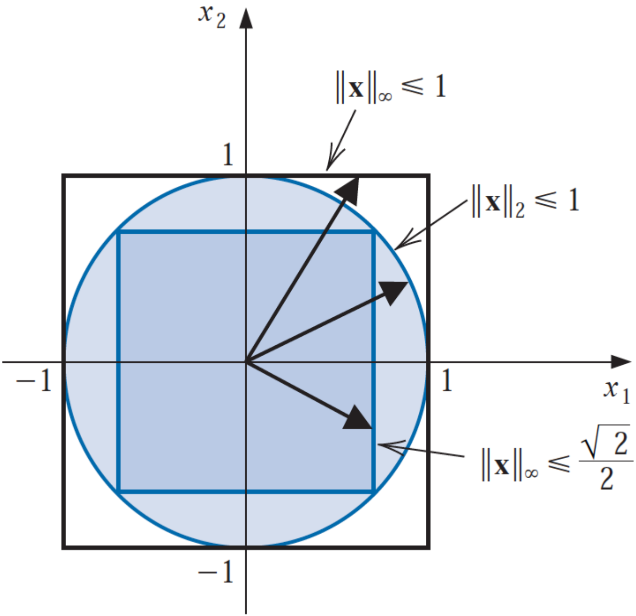
Theorem: For each $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$,

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\,\|\mathbf{x}\|_\infty. \qquad \square$$

Proof:

$$\|\mathbf{x}\|_\infty^2 \;=\; \mathbf{max}_{1\leq j\leq n}|x_j|^2 \leq \sum_{j=1}^{n}|x_j|^2 = \|\mathbf{x}\|_2^2 \leq \sum_{j=1}^{n}\|\mathbf{x}\|_\infty^2 = n\,\|\mathbf{x}\|_\infty^2.$$

Illustration: $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{2}\,\|\mathbf{x}\|_\infty$



Cauchy-Schwarz Inequality

Theorem: For each $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$,

$$\left|\mathbf{x}^T\mathbf{y}\right| = \left|\sum_{j=1}^{n} x_j\, y_j\right| \le \sqrt{\sum_{j=1}^{n} x_j^2}\,\sqrt{\sum_{j=1}^{n} y_j^2} = \|\mathbf{x}\|_2\ \|\mathbf{y}\|_2\,. \qquad \square$$

Example: for $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,

$$\begin{aligned} \|\mathbf{x}\|_2 &= \sqrt{14}, \quad \|\mathbf{y}\|_2 = \sqrt{3}, \\ \left|\mathbf{x}^T\mathbf{y}\right| &= 6 < \sqrt{14} \times \sqrt{3}. \end{aligned}$$

Proof of $\left|\sum_{j=1}^n x_j\, y_j\right| \le \sqrt{\sum_{j=1}^n x_j^2}\,\sqrt{\sum_{j=1}^n y_j^2}$

$$
\begin{aligned}
\left|\sum_{j=1}^n x_j\, y_j\right|^2 &= \left(\sum_{i=1}^n x_i\, y_i\right) \times \left(\sum_{j=1}^n x_j\, y_j\right) = \frac{1}{2}\sum_{i,j=1}^n \left(2\, x_i\, y_i,\, x_j\, y_j\right) \\
&= \frac{1}{2}\sum_{i,j=1}^n \Big(2\, x_i\, y_i,\, x_j\, y_j - (\,x_i\, y_j)^2 - (\,x_j\, y_i)^2\Big) \\
&\qquad + \Big((\,x_i\, y_j)^2 + (\,x_j\, y_i)^2\Big) \\
&= \frac{1}{2}\sum_{i,j=1}^n \Big(-(\,x_i\, y_j - x_j\, y_i)^2\Big) + \Big((\,x_i\, y_j)^2 + (\,x_j\, y_i)^2\Big) \\
&\le \frac{1}{2}\sum_{i,j=1}^n \Big((\,x_i\, y_j)^2 + (\,x_j\, y_i)^2\Big) = \left(\sum_{i=1}^n x_i^2\right) \times \left(\sum_{j=1}^n y_j^2\right).
\end{aligned}
$$

The Triangle Inequality

Theorem: For each $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$,

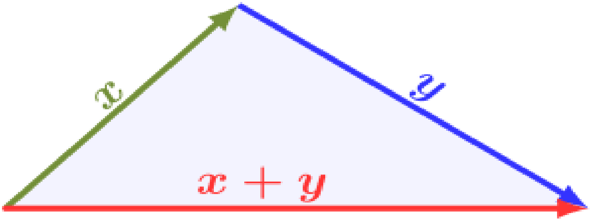
$$\|\mathbf{x}+\mathbf{y}\|_2 \leq \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2 \qquad \square$$

Example: for $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,

$$\begin{aligned} \|\mathbf{x}\|_2 &= \sqrt{14}, \quad \|\mathbf{y}\|_2 = \sqrt{3}, \\ \|\mathbf{x}+\mathbf{y}\|_2 &= \sqrt{29} = 5.38\cdots \\ &< \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2 = \sqrt{14} + \sqrt{3} = 5.47\cdots . \end{aligned}$$

Proof of Triangle Inequality: $\|\mathbf{x}+\mathbf{y}\|_2 \le \|\mathbf{x}\|_2 + \|\mathbf{y}\|_2$

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|_2^2 &= (\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y}) = \mathbf{x}^T\mathbf{x} + 2\,\mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{y} \\
&= \|\mathbf{x}\|_2^2 + 2\,\mathbf{x}^T\mathbf{y} + \|\mathbf{y}\|_2^2 \\
&\le \|\mathbf{x}\|_2^2 + 2\,\|\mathbf{x}\|_2\,\|\mathbf{y}\|_2 + \|\mathbf{y}\|_2^2 \\
&= (\|\mathbf{x}\|_2 + \|\mathbf{y}\|_2)^2 .
\end{aligned}$$



2−norm **Distance** and ∞−norm **Distance** for

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

Definition:

$$\|\mathbf{x}-\mathbf{y}\|_2 \stackrel{def}{=} \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2},$$

$$\|\mathbf{x}-\mathbf{y}\|_\infty \stackrel{def}{=} \mathbf{max}_{1\le j\le n}|x_j-y_j|.$$

$$\text{Let } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathbf{x}^{(k)} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix} \in \mathbb{R}^n \text{ for all } k \geq 1$$

Def: Sequence $\{\mathbf{x}^{(k)}\}$ is said to **converge** to $\mathbf{x}$ with respect to norm $\|\cdot\|$ if, given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \epsilon \quad \text{for all} \quad k \geq N(\epsilon).$$

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{x}^{(k)} = \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix} \in \mathbb{R}^n$ for all $k \geq 1$

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Thm: Sequence $\{\mathbf{x}^{(k)}\}$ is said to **converge** to $\mathbf{x}$ with respect to $\infty-$norm if and only if

$$\lim_{k\to\infty} x_i^{(k)} = x_i \quad \text{for each} \quad i = 1, \cdots, n.$$

Proof: $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\infty-$norm if and only if $\mathbf{lim}_{k\to\infty} x_i^{(k)} = x_i$ for each i

Assume $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\infty-$norm. Given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \epsilon \quad \text{for all} \quad k \geq N(\epsilon).$$

Thus for each $i = 1, \cdots, n$ and each $k \geq N(\epsilon)$,

$$|\mathbf{x}_i^{(k)} - \mathbf{x}_i| \leq \|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \epsilon.$$

By definition of limit, for each i

$$\mathbf{lim}_{k\to\infty} x_i^{(k)} = x_i.$$

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Conversely, assume for each i

$$\mathbf{lim}_{k\to\infty} x_i^{(k)} = x_i \quad \cdots \quad \text{Proof omitted.}$$

Proof: $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\infty-$norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$ for each i

Assume $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\infty-$norm. Given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\|_\infty < \epsilon \quad \text{for all} \quad k \geq N(\epsilon).$$

Thus for each $i = 1, \cdots, n$ and each $k \geq N(\epsilon)$,

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Conversely, assume for each i

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Ex: Sequence $\{\mathbf{x}^{(k)}\}$, $\mathbf{x}^{(k)} = \begin{pmatrix} 1 \\ 1/k \\ \sin(k)/k^2 \end{pmatrix}$, converges to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Matrix Norm

A MATRIX NORM on $\mathbb{R}^{n\times n}$ is a function, $\|\cdot\|$, from $\mathbb{R}^{n\times n}$ into $\mathbb{R}$ with the following properties:

(i) $\|A\| \geq 0$ for all $A \in \mathbb{R}^{n\times n}$,

(ii) $\|A\| = 0$ if and only if $A = \mathbf{0} \in \mathbb{R}^{n\times n}$,

(iii) $\|\alpha A\| = |\alpha|\,\|A\|$ for all $\alpha \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times n}$,

(iv) $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$,

(v) $\|A\,B\| \leq \|A\|\,\|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$.

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NATURAL NORM **Thm**: If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| \stackrel{def}{=} \mathbf{max}_{\mathbf{z}\neq 0} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|}$$

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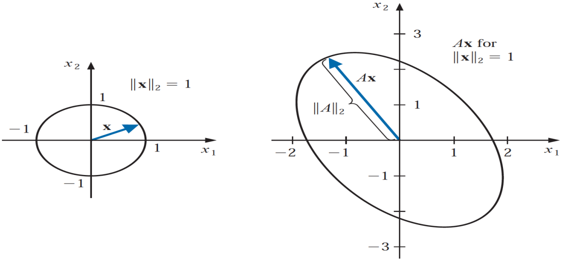
$$\|A\| \overset{def}{=} \mathbf{max}_{\mathbf{z}\neq 0} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|} \left(= \mathbf{max}_{\|\mathbf{z}\|=1} \|Az\|\right)$$

is a matrix norm. $\square$

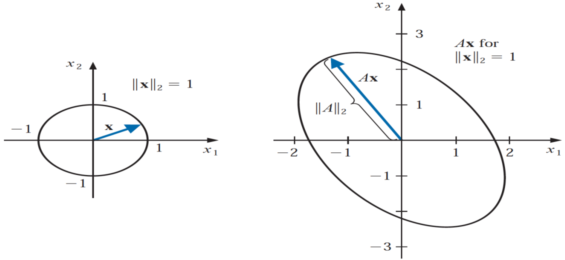
Matrix 2−norm and ∞−norm, $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

- Matrix 2−norm: $\|A\|_2 \stackrel{def}{=} \max_{\|z\|_2=1} \|Az\|_2$

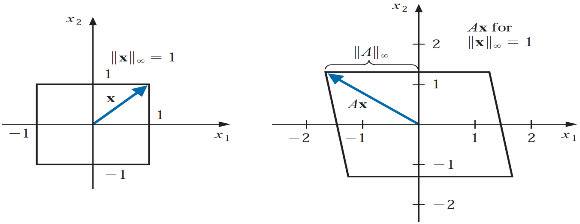


Matrix 2−norm and ∞−norm, $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

- Matrix 2−norm: $\|A\|_2 \stackrel{def}{=} \max_{\|z\|_2=1} \|Az\|_2$



- Matrix ∞−norm: $\|A\|_\infty \stackrel{def}{=} \max_{\|z\|_\infty=1} \|Az\|_\infty$



Thm: Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ then $\|A\|_\infty = \mathbf{max}_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

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Proof (Part I): Partition and define

$$A = \begin{pmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix},$$

and $\|\mathbf{a}_i\|_1 = \sum_{j=1}^n |a_{ij}|$ for $1 \le i \le n$. Then

$$A\,\mathbf{z} = \begin{pmatrix} \mathbf{a}_1^T\,\mathbf{z} \\ \vdots \\ \mathbf{a}_n^T\,\mathbf{z} \end{pmatrix}, \quad \text{therefore}$$

$$\begin{aligned} \|A\,\mathbf{z}\|_\infty &= \max_{1\le i\le n} \left|\mathbf{a}_i^T\,\mathbf{z}\right| \\ &\le \max_{1\le i\le n} \|\mathbf{a}_i\|_1 \,\|\mathbf{z}\|_\infty = \left(\max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|\right) \|\mathbf{z}\|_\infty . \end{aligned}$$

It follows that $\|A\|_\infty \le \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$.

Thm: Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ then $\|A\|_\infty = \mathbf{max}_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

Thm: Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ then $\|A\|_\infty = \mathbf{max}_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$

Proof (Part II): Let

$$\sum_{j=1}^{n} |a_{\imath,j}| = \mathbf{max}_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

and $\mathbf{z} = \mathbf{sign}\left(\mathbf{a}_\imath\right)$. Then $\|\mathbf{z}\|_\infty = 1$, and

$$\begin{aligned} \|A\,\mathbf{z}\|_\infty &\ge \left|\mathbf{a}_\imath^T\,\mathbf{z}\right| = \|\mathbf{a}_\imath\|_1 \\ &= \mathbf{max}_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|\,. \end{aligned}$$

Put together

$$\mathbf{max}_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \le \|A\|_\infty \le \mathbf{max}_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|\,.$$

Example: Matrix $\infty-$norm, $A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 1 \\ -2 & 1 & 0 \end{pmatrix}$.

- $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, and $\|\mathbf{a}_1\|_1 = 6$,
- $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $\|\mathbf{a}_2\|_1 = 3$,
- $\mathbf{a}_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, and $\|\mathbf{a}_3\|_1 = 3$,
- $$\|A\|_\infty = \mathbf{max}\,(6, 3, 3) = 6.$$

§7.2 Eigenvalues and Eigenvectors

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

- The CHARACTERISTIC POLYNOMIAL of A is defined by

$$p(\lambda) = \mathbf{det}(A - \lambda I).$$

- The EIGENVALUES of A are those values of λ such that

$$p(\lambda) = 0,$$

  i.e., those values of λ such that the matrix $A - \lambda I$ is singular.

- For any eigenvalue λ, its EIGENVECTOR $\mathbf{x}$ is any non-zero vector such that

$$(A - \lambda I)\mathbf{x} = 0.$$

Ex: Eigenvalues/Eigenvectors of $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}$.

- The CHARACTERISTIC POLYNOMIAL of A is

$$\begin{aligned} p(\lambda) &= \textbf{det}\,(A - \lambda I) = \textbf{det} \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 1 & -1 & 4-\lambda \end{pmatrix} \\ &= (2-\lambda)\,\textbf{det} \begin{pmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix} = (2-\lambda)\left(\lambda^2 - 5\lambda + 6\right) \\ &= -(\lambda-2)^2(\lambda-3). \end{aligned}$$

- For eigenvalue $\lambda_1 = 3$, its EIGENVECTOR $\mathbf{x}_1$ satisfies

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x}_1 = 0, \quad \text{implying} \quad \mathbf{x}_1 = \begin{pmatrix} 0 \\ \xi \\ \xi \end{pmatrix} \quad \text{for } \xi \neq 0.$$

Ex: Eigenvalues/Eigenvectors of $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}$.

- For double eigenvalue $\lambda_2 = 2$, its EIGENVECTOR $\mathbf{x}_2$ satisfies

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix} \mathbf{x}_2 = 0, \quad \text{i.e.,} \quad \mathbf{x}_2 = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \neq 0.$$

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

- The SPECTRAL RADIUS of A is defined by

$$\rho(A) \stackrel{def}{=} \mathbf{max}\{|\lambda| \mid \lambda \text{ is an eigenvalue of } A.\}$$

- **Thm**:
  - $\|A\|_2 = \left(\rho\left(A^T A\right)\right)^{\frac{1}{2}}$.
  - $\rho(A) \leq \|A\|$ for any natural norm $\|\cdot\|$.

Ex: Find 2−norm of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

Solution: By earlier theorem,

$$\|A\|_2 = \left(\rho\left(A^T A\right)\right)^{\frac{1}{2}}.$$

Ex: Find 2−norm of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

Solution: By earlier theorem,

$$\|A\|_2 = \left(\rho\left(A^T A\right)\right)^{\frac{1}{2}}.$$

Calculating,

$$A^T A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{pmatrix},$$

Ex: Find 2−norm of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

Solution: By earlier theorem,

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and

$$\begin{aligned} \mathbf{det}\left(A^T A - \lambda I\right) &= \mathbf{det}\begin{pmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{pmatrix} \\ &= -\lambda^3 + 14\lambda^2 - 42\lambda. \end{aligned}$$

Ex: Find 2−norm of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$

Solution: By earlier theorem,

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Calculating,

$$A^T A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{pmatrix},$$

$$\text{and} \quad \mathbf{det}\left(A^T A - \lambda I\right) = \mathbf{det} \begin{pmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{pmatrix}$$

$$= -\lambda^3 + 14\lambda^2 - 42\lambda.$$

Setting $\mathbf{det}\left(A^T A - \lambda I\right) = 0$ leads to $\lambda = 0, 7 \pm \sqrt{7}$. Thus

$$\|A\|_2 = \sqrt{\mathbf{max}\left\{0, 7-\sqrt{7}, 7+\sqrt{7}\right\}} = \sqrt{7+\sqrt{7}}.$$

Convergent Matrices

A matrix $A \in \mathbb{R}^{n\times n}$ is CONVERGENT if

$$\lim_{k\to\infty} \left(A^k\right)_{i,j} = 0, \quad \text{for all } 1 \le i, j \le n.$$

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Ex: Show that

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 5 & \frac{1}{2} \end{pmatrix} \quad \text{is a convergent matrix.}$$

Solution: Easy to verify that

$$A^2 = \begin{pmatrix} \frac{1}{2^2} & 0 \\ \frac{20}{2^2} & \frac{1}{2^2} \end{pmatrix}, \quad A^k = \begin{pmatrix} \frac{1}{2^k} & 0 \\ \frac{10\,k}{2^k} & \frac{1}{2^k} \end{pmatrix} \quad \text{for all } k \ge 1.$$

Thm: The following statements are equivalent

(i) $A \in \mathbb{R}^{n\times n}$ is a convergent matrix,

(ii) $\lim_{k\to\infty} \left\|A^k\right\| = 0$ for some natural norm,

(iii) $\lim_{k\to\infty} \left\|A^k\right\| = 0$ for all natural norms,

(iv) $\rho(A) < 1$,

(v) $\lim_{k\to\infty} A^k \mathbf{x} = 0$ for every $\mathbf{x} \in \mathbb{R}^n$.

§7.3 The Jacobi and Gauss-Siedel Iterative Techniques

- **Problem:** To solve $A\mathbf{x} = \mathbf{b}$ for $A \in \mathbb{R}^{n\times n}$.
- **Methodology:** Iteratively approximate solution $\mathbf{x}$. No GEPP.

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MATRIX SPLITTING

$$A = \mathbf{diag}\left(a_{1,1}, a_{2,2}, \cdots, a_{n,n}\right) + \begin{pmatrix} 0 & & & & \\ a_{2,1} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & 0 & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ & 0 & \cdots & a_{2,n-1} & a_{2,n} \\ & & \ddots & \vdots & \vdots \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix}$$

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- **Problem:** To solve $A\mathbf{x} = \mathbf{b}$ for $A \in \mathbb{R}^{n \times n}$.
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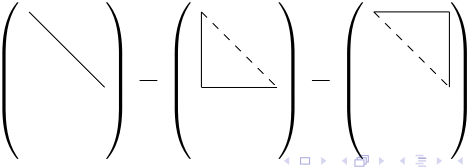
Matrix splitting

$$A = \mathbf{diag}\left(a_{1,1}, a_{2,2}, \cdots, a_{n,n}\right) + \begin{pmatrix} 0 & & & & \\ a_{2,1} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & 0 & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ & 0 & \cdots & a_{2,n-1} & a_{2,n} \\ & & \ddots & \vdots & \vdots \\ & & & 0 & a_{n-1,n} \\ & & & & 0 \end{pmatrix}$$

$$\stackrel{def}{=} D - L - U =$$



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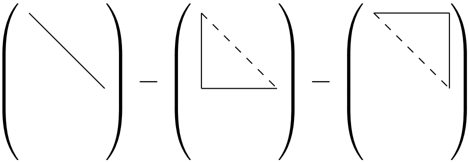
Ex: Matrix splitting for $A = \begin{pmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{pmatrix}$

$A =$ 

$$= \mathbf{diag}\,(10, 11, 10, 8) - \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ -2 & 1 & 0 & \\ 0 & -3 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -2 & 0 \\ & 0 & 1 & -3 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

The Jacobi and Gauss-Siedel Methods for solving $A\mathbf{x} = \mathbf{b}$

JACOBI METHOD: With matrix splitting $A = D - L - U$, rewrite

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}.$$

Jacobi iteration with given $\mathbf{x}^{(0)}$,

$$\mathbf{x}^{(k+1)} = D^{-1}(L + U)\mathbf{x}^{(k)} + D^{-1}\mathbf{b}, \quad \text{for} \quad k = 0, 1, 2, \cdots.$$

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GAUSS-SIEDEL METHOD: Rewrite

$$\mathbf{x} = (D-L)^{-1}\, U\mathbf{x} + (D-L)^{-1}\,\mathbf{b}.$$

Gauss-Siedel iteration with given $\mathbf{x}^{(0)}$,

$$\mathbf{x}^{(k+1)} = (D-L)^{-1}\, U\mathbf{x}^{(k)} + (D-L)^{-1}\,\mathbf{b}, \quad \text{for} \quad k = 0, 1, 2, \cdots.$$

Ex: Jacobi Method for $A\mathbf{x} = \mathbf{b}$, with

$$A = \begin{pmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 25 \\ -11 \\ 15 \end{pmatrix}$$

$$\begin{aligned} A &= D - L - U \\ &= \mathbf{diag}\,(10, 11, 10, 8) - \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ -2 & 1 & 0 & \\ 0 & -3 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -2 & 0 \\ & 0 & 1 & -3 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \end{aligned}$$

Ex: Jacobi Method for $A\mathbf{x} = \mathbf{b}$, with

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Jacobi iteration with $\mathbf{x}^{(0)} = \mathbf{0}$, for $k = 0, 1, 2, \cdots$

$$\begin{aligned} \mathbf{x}_{\mathbf{J}}^{(k+1)} &= D^{-1}(L + U)\,\mathbf{x}_{\mathbf{J}}^{(k)} + D^{-1}\,\mathbf{b} \\ &= \begin{pmatrix} 0 & \frac{1}{10} & -\frac{2}{10} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{2}{10} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{pmatrix} \mathbf{x}_{\mathbf{J}}^{(k)} + \begin{pmatrix} \frac{6}{10} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{pmatrix} \end{aligned}$$

Ex: Gauss-Siedel Method for $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned} A &= D - L - U \\ &= \begin{pmatrix} 10 & & & \\ -1 & 11 & & \\ 2 & -1 & 10 & \\ 0 & 3 & -1 & 8 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -2 & 0 \\ & 0 & 1 & -3 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}. \end{aligned}$$

Ex: Gauss-Siedel Method for $A\mathbf{x} = \mathbf{b}$

$$\begin{aligned} A &= D - L - U \\ &= \begin{pmatrix} 10 & & & \\ -1 & 11 & & \\ 2 & -1 & 10 & \\ 0 & 3 & -1 & 8 \end{pmatrix} - \begin{pmatrix} 0 & 1 & -2 & 0 \\ & 0 & 1 & -3 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}. \end{aligned}$$

Gauss-Siedel iteration with $\mathbf{x}^{(0)} = \mathbf{0}$, for $k = 0, 1, 2, \cdots$

$$\begin{aligned} \mathbf{x}_{\mathbf{GS}}^{(k+1)} &= (D - L)^{-1}\, U\,\mathbf{x}_{\mathbf{GS}} + (D - L)^{-1}\,\mathbf{b} \\ &= \begin{pmatrix} 10 & & & \\ -1 & 11 & & \\ 2 & -1 & 10 & \\ 0 & 3 & -1 & 8 \end{pmatrix}^{-1} \left(\begin{pmatrix} 0 & 1 & -2 & 0 \\ & 0 & 1 & -3 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \mathbf{x}_{\mathbf{GS}}^{(k)} \right) \\ &\quad + \begin{pmatrix} \frac{6}{10} \\ \frac{25}{11} \\ -\frac{11}{10} \\ \frac{15}{8} \end{pmatrix}. \end{aligned}$$

General Iteration Methods

To solve $A\mathbf{x} = \mathbf{b}$ with matrix splitting $A = D - L - U$,

- JACOBI METHOD:

$$\mathbf{x}_{\mathbf{J}}^{(k+1)} = D^{-1}(L+U)\,\mathbf{x}_{\mathbf{J}}^{(k)} + D^{-1}\,\mathbf{b}.$$

- GAUSS-SIEDEL METHOD:

$$\mathbf{x}_{\mathbf{GS}}^{(k+1)} = (D-L)^{-1}\;U\,\mathbf{x}_{\mathbf{GS}}^{(k)} + (D-L)^{-1}\;\mathbf{b}.$$

GENERAL ITERATION METHOD: for $k = 0, 1, 2, \cdots$

$$\mathbf{x}^{(k+1)} = T\,\mathbf{x}^{(k)} + \mathbf{c}.$$

Next: convergence analysis on General Iteration Method

General Iteration: $\mathbf{x}^{(k+1)} = T\,\mathbf{x}^{(k)} + \mathbf{c}$ for $k = 0, 1, 2, \cdots$

Thm: The following statements are equivalent

- $\rho(T) < 1$.
- The equation

$$\mathbf{x} = T\,\mathbf{x} + \mathbf{c} \qquad (1)$$

  has a unique solution and $\{\mathbf{x}^{(k)}\}$ converges to this solution from any $\mathbf{x}^{(0)}$.

General Iteration: $\mathbf{x}^{(k+1)} = T\,\mathbf{x}^{(k)} + \mathbf{c}$ for $k = 0, 1, 2, \cdots$

Thm: The following statements are equivalent

- $\rho(T) < 1$.
- The equation
  $$\mathbf{x} = T\,\mathbf{x} + \mathbf{c} \qquad (1)$$
  has a unique solution and $\{\mathbf{x}^{(k)}\}$ converges to this solution from any $\mathbf{x}^{(0)}$.

Proof: Assume $\rho(T) < 1$. Then (1) has unique solution $\mathbf{x}^{(*)}$.

$$\begin{aligned}\mathbf{x}^{(k+1)} - \mathbf{x}^{(*)} &= T\left(\mathbf{x}^{(k)} - \mathbf{x}^{(*)}\right) = T^2\left(\mathbf{x}^{(k-1)} - \mathbf{x}^{(*)}\right) \\ &= \cdots = T^{k+1}\left(\mathbf{x}^{(0)} - \mathbf{x}^{(*)}\right) \Longrightarrow \mathbf{0}.\end{aligned}$$

Conversely, if $\cdots$ (omitted)