

## Gaussian Quadrature with Legendre polynomials

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where  $x_1, x_2, \dots, x_n \in (-1, 1)$  are distinct roots of  $P_n(x)$ .

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$$c_i \stackrel{\text{def}}{=} \int_{-1}^1 L_i(x) dx = \int_{-1}^1 \left( \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \right) dx \quad \text{for } i = 1, \dots, n.$$

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- ▶ Next: Estimate error in quadrature

## Hermite Interpolation, with Legendre polynomial $P_n(x)$

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$$(x_1, f(x_1), f'(x_1)), (x_2, f(x_2), f'(x_2)), \dots, (x_n, f(x_n), f'(x_n)),$$

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- ▶ **Theorem:** For each  $x \in [a, b]$ , a number  $\xi(x)$  between  $x_1, x_2, \dots, x_n$  (hence  $\in (a, b)$ ) exists with

$$f(x) = H(x) + R(x), \quad R(x) = \frac{f^{(2n)}(\xi(x))}{(2n)!} (x-x_1)^2 (x-x_2)^2 \cdots (x-x_n)^2.$$

## Hermite Interpolation, with Gaussian Quadrature

$$\int_{-1}^1 f(x) dx = \int_{-1}^1 H(x) dx + \int_{-1}^1 R(x) dx$$



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## Gaussian Quadrature Error Estimate

$$\mathbf{R} = \frac{f^{(2n)}(\xi)}{(2n)!} \int_{-1}^1 (x - x_1)^2 (x - x_2)^2 \cdots (x - x_n)^2 dx$$

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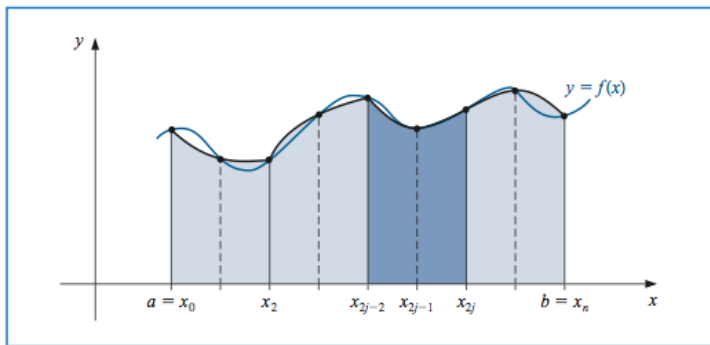
$$\int_{-1}^1 f(x) dx = c_1 f(x_1) + c_2 f(x_2) + \cdots + c_n f(x_n) + O\left(\frac{4^{-n} |f^{(2n)}(\xi)|}{(2n)!}\right).$$

Rapid convergence for smooth functions

## Composite Simpson's Rule

$$(n = 2m, x_j = a + jh, h = \frac{b-a}{n}, 0 \leq j \leq n)$$

$$\int_a^b f(x) dx = \frac{h}{3} \left( f(a) + 2 \sum_{j=1}^{m-1} f(x_{2j}) + 4 \sum_{j=1}^m f(x_{2j-1}) + f(b) \right) - \frac{(b-a)h^4}{180} f^{(4)}(\mu)$$





# Limitations of Gaussian Quadrature

Simpson/Trapezoidal:

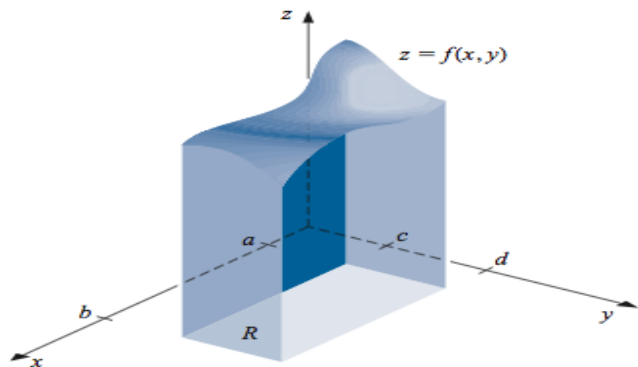
- ▶ Composite rules:
  - ▶ Adding more `EQUI-SPACED` points.
- ▶ Romberg extrapolation:
  - ▶ Obtaining higher order rules from lower order rules.
- ▶ Adaptive quadratures:
  - ▶ Adding more points `ONLY WHEN NECESSARY`.

Gaussian Quadrature:

- ▶ points different for different  $n$ .

Gaussian Quadrature good for given  $n$ ,  
not as good for given tolerance.

# Double Integral $\iint_R f(x, y) dA$



$$R = \{(x, y) \mid a \leq x \leq b, \quad c \leq y \leq d.\}$$

## Double Integral = Integral of Integral

$$\begin{aligned}\iint_R f(x, y) dA &= \int_a^b \left( \int_c^d f(x, y) dy \right) dx \\ &= \int_a^b g(x) dx,\end{aligned}$$

where  $g(x) \stackrel{\text{def}}{=} \int_c^d f(x, y) dy$ .

Approach

- ▶ Approximate  $\int_a^b g(x) dx$  with quadrature.
- ▶ For any given  $x_i$ , approximate  $g(x_i)$  with quadrature.

$$\int \int_R f(x, y) dA = \int_a^b g(x) dx, \quad g(x) = \int_c^d f(x, y) dy$$

- ▶ Approximate  $\int_a^b g(x) dx$  with  $n$ -point quadrature:

$$\int_a^b g(x) dx = c_1 g(x_1) + c_2 g(x_2) + \cdots + c_n g(x_n) + \mathbf{R}(g)$$

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- ▶ For  $1 \leq i \leq n$ , approximate  $g(x_i)$  with  $m$ -point quadrature:

$$\int_c^d f(x_i, y) dy = \hat{c}_1 f(x_i, y_1) + \hat{c}_2 f(x_i, y_2) + \cdots + \hat{c}_m f(x_i, y_m) + \hat{\mathbf{R}}(f(x_i, \cdot)).$$

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$$\begin{aligned} \int \int_R f(x, y) dA &= \left( \sum_{i=1}^n c_i g(x_i) \right) + \mathbf{R}(g) \\ &= \left( \sum_{i=1}^n c_i \left( \left( \sum_{j=1}^m \hat{c}_j f(x_i, y_j) \right) + \hat{\mathbf{R}}(f(x_i, \cdot)) \right) \right) + \mathbf{R}(g) \end{aligned}$$

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$$\int \int_R f(x, y) dA = \left( \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j) \right) + \left( \sum_{i=1}^n c_i \hat{\mathbf{R}}(f(x_i, \cdot)) \right) + \mathbf{R}(g) \\ \approx \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j).$$

- ▶ Double integral quadrature is a double sum.
- ▶ Need to work out total error for any given quadrature.

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**Example**, Simpson's Rule with  $m = n = 3$ :

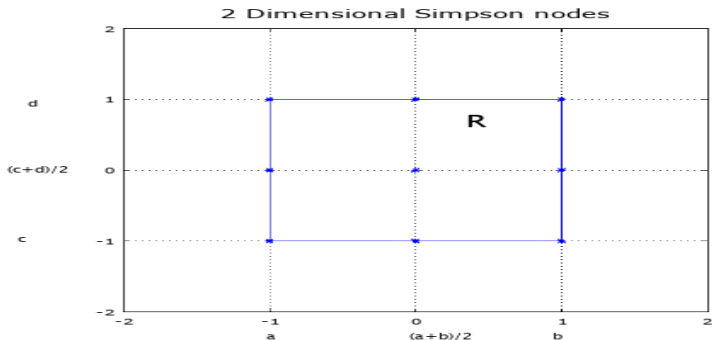
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$$\int_a^b g(x) dx = c_1 g(x_1) + c_2 g(x_2) + c_3 g(x_3) - \frac{h^5}{90} g^{(4)}(\xi),$$

$$h = \frac{b-a}{2}, \quad (c_1, c_2, c_3) = \frac{h}{3} (1, 4, 1).$$

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$$k = \frac{d-c}{2}, \quad (\hat{c}_1, \hat{c}_2, \hat{c}_3) = \frac{k}{3} (1, 4, 1).$$

$$\int \int_R f(x, y) dA = \left( \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j) \right) - \frac{k^5}{90} \left( \sum_{i=1}^m c_i \frac{\partial^4 f}{\partial^4 y} (x_i, \eta_i) \right) \\ - \frac{h^5}{90} \int_a^b \frac{\partial^4 f}{\partial^4 x} (\xi, y) dy$$

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&\quad - \frac{h^5}{90} (b - a) \frac{\partial^4 f}{\partial^4 x}(\xi, \eta) \\
&= \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j) \\
&\quad - \frac{(b - a)(d - c)}{180} \left( k^4 \frac{\partial^4 f}{\partial^4 y}(\hat{\xi}, \hat{\eta}) + h^4 \frac{\partial^4 f}{\partial^4 x}(\xi, \eta) \right).
\end{aligned}$$

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**Example:** Composite Simpson Rules.



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- ▶ Composite Simpson on  $[a, b]$ ,  $x_i = a + (i-1)h$ ,  $1 \leq i \leq n$ ,  $h = \frac{b-a}{n-1}$ .

$$\mathbf{R}(g) = -\frac{(b-a)h^4}{180} \int_c^d \frac{\partial^4 f}{\partial^4 x}(\xi, y) dy = -\frac{(b-a)(d-c)h^4}{180} \frac{\partial^4 f}{\partial^4 x}(\xi, \eta).$$

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**Example:** Composite Simpson Rules.

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$$\mathbf{R}(g) = -\frac{(b-a)h^4}{180} \int_c^d \frac{\partial^4 f}{\partial^4 x}(\xi, y) dy = -\frac{(b-a)(d-c)h^4}{180} \frac{\partial^4 f}{\partial^4 x}(\xi, \eta).$$

- ▶ Composite Simpson on  $[c, d]$ ,  $y_j = c + (j-1)k$ ,  $1 \leq j \leq m$ ,  $k = \frac{d-c}{m-1}$ .

$$\begin{aligned} \hat{\mathbf{R}}(f(x_i, \cdot)) &= -\frac{(d-c)k^4}{180} \frac{\partial^4 f}{\partial^4 y}(x_i, \eta_i) \\ \sum_{i=1}^n c_i \hat{\mathbf{R}}(f(x_i, \cdot)) &= -\frac{(d-c)k^4}{180} \left( \sum_{i=1}^n c_i \right) \frac{\partial^4 f}{\partial^4 y}(\hat{\xi}, \hat{\eta}) \\ &= -\frac{(b-a)(d-c)k^4}{180} \frac{\partial^4 f}{\partial^4 y}(\hat{\xi}, \hat{\eta}). \end{aligned}$$

## Error Estimate, Double Integral with Composite Simpson

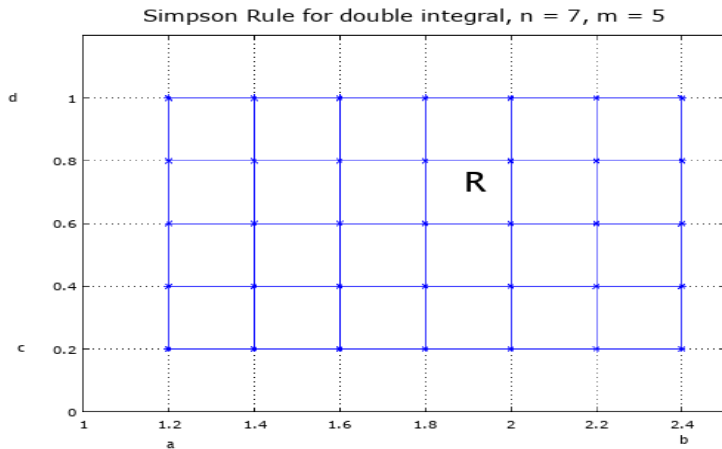
$$R = \{(x, y) \mid a \leq x \leq b, \quad c \leq y \leq d.\}$$

$$\int \int_R f(x, y) dA = \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j) - \frac{(b-a)(d-c)}{180} \left( k^4 \frac{\partial^4 f}{\partial^4 y}(\hat{\xi}, \hat{\eta}) + h^4 \frac{\partial^4 f}{\partial^4 x}(\xi, \eta) \right),$$

$$h = \frac{b-a}{n-1}, \quad k = \frac{d-c}{m-1}.$$

Example:  $\int \int_R \log(x + 2y) dA$  with  $n = 7, m = 5$

$$R = \{(x, y) \mid 1.2 \leq x \leq 2.4, \quad 0.2 \leq y \leq 1.\}$$



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$$\left| \frac{\partial^4 f}{\partial^4 x} \right| \leq \frac{6}{(1.2 + 2 \times 0.2)^4} \approx 0.91553 \quad \text{for } (x, y) \in R,$$

$$\left| \frac{\partial^4 f}{\partial^4 y} \right| \leq \frac{96}{(1.2 + 2 \times 0.2)^4} \approx 14.648.$$

$$\begin{aligned} \mathbf{Quad\_Error} &= \frac{(b - a)(d - c)}{180} \left| k^4 \frac{\partial^4 f}{\partial^4 y}(\hat{\xi}, \hat{\eta}) + h^4 \frac{\partial^4 f}{\partial^4 x}(\xi, \eta) \right| \\ &\leq \frac{(2.4 - 1.2)(1 - 0.2)}{180} (0.2^4 \times 14.648 + 0.2^4 \times 0.91553) \end{aligned}$$

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$$\begin{aligned} \int \int_R \mathbf{log}(x + 2y) dA &\approx \sum_{i=1}^7 \sum_{j=1}^5 c_i \hat{c}_j f(x_i, y_j) \\ &= 1.0360327 \dots \end{aligned}$$

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Therefore

$$\left| \int \int_R \mathbf{log}(x + 2y) dA - \left( \sum_{i=1}^7 \sum_{j=1}^5 c_i \hat{c}_j f(x_i, y_j) \right) \right| = 1.546 \times 10^{-5}$$

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## 2 Dimensional Gaussian Quadratures

$$R = \{(x, y) \mid a \leq x \leq b, \quad c \leq y \leq d.\}$$
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Perform change of variables

$$x = \frac{a+b}{2} + \frac{b-a}{2} u, \quad y = \frac{c+d}{2} + \frac{d-c}{2} v \quad \text{for } u, v \in [-1, 1].$$

Double integral becomes

$$\int \int_R f(x, y) dA = \frac{(b-a)(d-c)}{4} \int_{-1}^1 \hat{g}(u) du, \quad \text{where}$$

$$\hat{g}(u) \stackrel{\text{def}}{=} \int_{-1}^1 f \left( \frac{a+b}{2} + \frac{b-a}{2} u, \frac{c+d}{2} + \frac{d-c}{2} v \right) dv.$$

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- ▶  $m$ -point Gaussian quadrature for  $\widehat{g}(u_i)$ ,

$$\begin{aligned} \widehat{g}(u_i) &\approx \widehat{c}_1 f \left( x_i, \frac{c+d}{2} + \frac{d-c}{2} v_1 \right) + \cdots + \widehat{c}_m f \left( x_i, \frac{c+d}{2} + \frac{d-c}{2} v_m \right) \\ &= \widehat{c}_1 f(x_i, y_1) + \cdots + \widehat{c}_m f(x_i, y_m), \quad y_j \stackrel{\text{def}}{=} \frac{c+d}{2} + \frac{d-c}{2} v_j. \end{aligned}$$

$$\int \int_R f(x, y) dA = \frac{(b-a)(d-c)}{4} \int_{-1}^1 \widehat{g}(u) du$$

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$$\widehat{g}(u_i) \approx \widehat{c}_1 f(x_i, y_1) + \cdots + \widehat{c}_m f(x_i, y_m).$$

So we have Gaussian quadrature for double integral:

$$\int \int_R f(x, y) dA \approx \frac{(b-a)(d-c)}{4} \sum_{i=1}^n \sum_{j=1}^m c_i \widehat{c}_j f(x_i, y_j).$$

Example:  $\int \int_R \mathbf{log}(x + 2y) dA = 1.036 \dots$ ,  $n = 7$ ,  $m = 5$

► Gaussian quadrature approximation

$$\int \int_R f(x, y) dA \approx \frac{(b-a)(d-c)}{4} \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j) \approx 1.03604817065 \dots$$

$$\left| \int \int_R f(x, y) dA - \frac{(b-a)(d-c)}{4} \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j) \right| \approx 6.4 \times 10^{-10}.$$

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$$\iint_R f(x, y) dA \approx \sum_{i=1}^n \sum_{j=1}^m c_i \hat{c}_j f(x_i, y_j) \approx 1.03603270963 \dots$$

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Gaussian quadrature much more accurate.

## Gaussian Quadrature: how good is it?

- ▶ Gaussian quadrature

$$\begin{aligned}\int_{-1}^1 f(x) dx &= c_1 f(x_1) + c_2 f(x_2) + \cdots + c_n f(x_n) \\ &\quad + O\left(\frac{4^{-n} |f^{(2n)}(\xi)|}{(2n)!}\right) \\ &\approx c_1 f(x_1) + c_2 f(x_2) + \cdots + c_n f(x_n),\end{aligned}$$

where  $x_1, x_2, \dots, x_n \in (-1, 1)$  are distinct roots of Legendre Polynomial  $P_n(x)$ .

- ▶ Error tiny for large  $n$  and  $|f^{(2n)}(\xi)| = O(1)$ .

**Improper Integral:** what if  $\max_{x \in [-1, 1]} |f^{(2n)}(x)| \gg 1$ ?

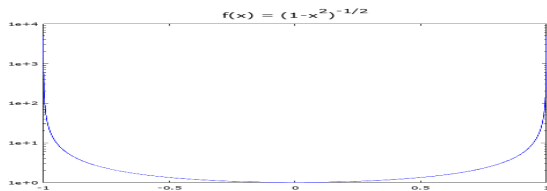
Gaussian Quadrature for  $\int_{-1}^1 f(x)dx$ ,  $f(x) = \frac{1}{\sqrt{1-x^2}}$

$n$	Quadrature Value
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250	3.1346
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- $f(x) = \frac{1}{\sqrt{1-x^2}}$  not smooth in  $[-1,1]$

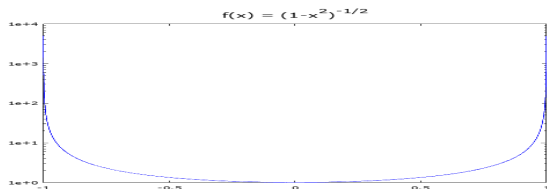




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$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \stackrel{x=\sin\theta}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 d\theta = 3.1416\dots$$

## Improper Integral: $\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx$

- ▶ Change of variable:

$$x = \mathbf{\sin} \theta, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$



$$dx = \mathbf{\cos} \theta, \quad \sqrt{1-x^2} = \sqrt{1-\mathbf{\sin}^2\theta} = \mathbf{\cos} \theta.$$

- ▶ New integral becomes proper.

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\mathbf{\sin} \theta) d\theta.$$

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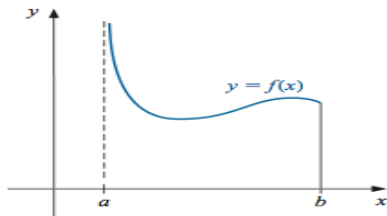
$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\mathbf{\sin} \theta) d\theta.$$

- ▶ **Example:** for  $f(x) = x^2$ , with 20-point Gaussian quadrature:

$$\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{\sin}^2 \theta d\theta \approx 1.57079632679490,$$

accurate to 15 digits.

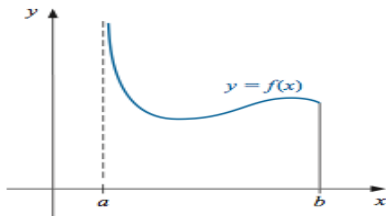
Improper Integral:  $\int_a^b f(x)dx$ ,  $f(x) = \frac{g(x)}{(x-a)^p}$ , for  $p < 1$



- ▶ Integral is improper if  $g(a) \neq 0$ :

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- ▶ Integral is not defined for  $p \geq 1$ :

$$\begin{aligned} \int_a^b \frac{1}{(x-a)^p} dx &= \lim_{M \rightarrow a^+} \int_M^b \frac{1}{(x-a)^p} dx \\ &= \lim_{M \rightarrow a^+} \left. \frac{(x-a)^{1-p}}{1-p} \right|_M^b = \frac{(b-a)^{1-p}}{1-p}. \end{aligned}$$

## Proper Method for improper Integral $\int_a^b \frac{g(x)}{(x-a)^p} dx$

- ▶ Assume a Taylor expansion on  $g(x)$ :

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \cdots + \frac{g^{(n)}(a)}{n!}(x-a)^n + \cdots$$

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$$P_k(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \cdots + \frac{g^{(k)}(a)}{k!}(x-a)^k.$$

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- ▶ First integral is a simple sum:

$$\int_a^b \frac{P_k(x)}{(x-a)^p} dx = \sum_{j=0}^k \int_a^b \frac{g^{(j)}(a)}{j!} \frac{(x-a)^j}{(x-a)^p} dx = \sum_{j=0}^k \frac{g^{(j)}(a)}{j!(j+1-p)} (b-a)^{j+1-p}.$$

## Proper Method for improper Integral $\int_a^b \frac{g(x)}{(x-a)^p} dx$

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- ▶ Second integral is a proper integral for  $k \gg p$ :

$$\int_a^b \frac{g(x) - P_k(x)}{(x-a)^p} dx = \int_a^b (x-a)^{k+1-p} \left( \sum_{j=k+1}^{\infty} \frac{g^{(j)}(a)}{j!} (x-a)^{j-k-1} \right) dx.$$

## Summary of Proper Method

- ▶ Choose a  $(k + 1)$ -term approximation:

$$P_k(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2}(x-a)^2 + \cdots + \frac{g^{(k)}(a)}{k!}(x-a)^k.$$

- ▶ Improper integral decomposition

$$\begin{aligned} \int_a^b \frac{g(x)}{(x-a)^p} dx &= \int_a^b \frac{P_k(x)}{(x-a)^p} dx + \int_a^b \frac{g(x) - P_k(x)}{(x-a)^p} dx. \\ &= \left( \sum_{j=0}^k \frac{g^{(j)}(a)}{j!(j+1-p)} (b-a)^{j+1-p} \right) + \int_a^b G(x) dx, \end{aligned}$$

$$\text{where } G(x) = \begin{cases} 0, & \text{if } x = a, \\ \frac{g(x) - P_k(x)}{(x-a)^p}. & \text{if } x > a. \end{cases}$$

Only require  $g(a), g'(a), \dots, g^{(k)}(a)$

Example:  $\int_0^1 \frac{e^x}{\sqrt{x}} dx$

- ▶ Take a 5-term Taylor expansion

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$



$$\begin{aligned} \int_0^1 \frac{P_4(x)}{\sqrt{x}} dx &= \int_0^1 \left( \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{2} + \frac{x^{5/2}}{6} + \frac{x^{7/2}}{24} \right) dx \\ &= 2 + \frac{2}{3} + \frac{1}{5} + \frac{1}{21} + \frac{1}{108} \\ &\approx 2.9235450. \end{aligned}$$

- ▶ Composite Simpson's rule with  $n = 4$ ,  $h = \frac{1}{n} = 0.25$  on

$$\int_0^1 G(x) dx, \quad \text{where } G(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{e^x - P_4(x)}{\sqrt{x}}. & \text{if } x > 0. \end{cases}$$

Example:  $\int_0^1 \frac{e^x}{\sqrt{x}} dx$

x	G(x)
0.0	0
0.25	0.0000170
0.50	0.0004013
0.75	0.0026026
1.0	0.0099485

- ▶ Composite Simpson rule, with  $n = 4$ ,  $h = \frac{1}{n} = 0.25$

$$\int_0^1 G(x) dx \approx \frac{0.25}{3} (0 + 4 \times 0.0000170 + 2 \times 0.0004013 + 4 \times 0.0026026 + 0.0099485) \approx 0.0017691.$$

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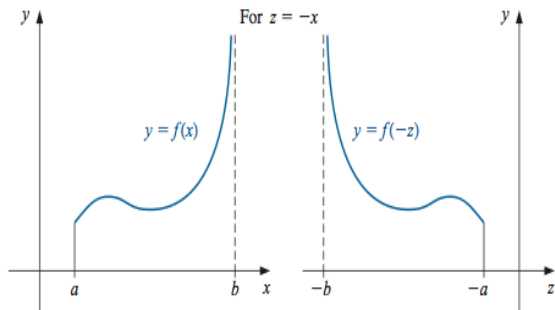
- ▶ Improper integral decomposition

$$\begin{aligned} \int_0^1 \frac{e^x}{\sqrt{x}} dx &= \int_0^1 \frac{P_4(x)}{\sqrt{x}} dx + \int_0^1 \frac{e^x - P_4(x)}{\sqrt{x}} dx \\ &\approx 2.9235450 + 0.0017691 = 2.9253141. \end{aligned}$$

Improper Integral:  $\int_a^b f(x)dx$ ,  $f(x) = \frac{g(x)}{(b-x)^p}$ , for  $p < 1$

- ▶ Change of variable:  $z = -x$
- ▶ Left endpoint improper Integral

$$\int_a^b \frac{g(x)}{(b-x)^p} dx = \int_{-b}^{-a} \frac{g(-z)}{(z - (-b))^p} dz$$



Improper Integral:  $\int_a^\infty f(x)dx$ ,  $f(x) = \frac{g(x)}{x^p}$ , for  $p > 1$

- ▶ Integral is not defined for  $p \leq 1$ :

$$\begin{aligned}\int_a^\infty \frac{1}{x^p} dx &= \lim_{M \rightarrow \infty} \int_a^M \frac{1}{x^p} dx \\ &= \lim_{M \rightarrow \infty} \left. \frac{x^{1-p}}{1-p} \right|_a^\infty = \frac{a^{1-p}}{p-1}.\end{aligned}$$

- ▶ In general, change of variable  $z = \frac{1}{x}$ , assuming  $a > 0$ .

$$dx = -\frac{dz}{z^2}.$$

- ▶ Left end improper integral:

$$\int_a^\infty \frac{g(x)}{x^p} dx = \int_0^{\frac{1}{a}} g\left(\frac{1}{z}\right) z^{p-2} dz.$$



Example:  $\int_1^{\infty} f(x) dx$ ,  $f(x) = \frac{\sin(\frac{1}{x})}{x^{3/2}}$

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$$\begin{aligned} \int_0^1 \frac{\sin(z)}{\sqrt{z}} dz &= \int_0^1 \frac{z - \frac{z^3}{6}}{\sqrt{z}} dz + \int_0^1 \frac{\sin(z) - \left(z - \frac{z^3}{6}\right)}{\sqrt{z}} dz \\ &\approx 0.61904761 + \int_0^1 \frac{\sin(z) - \left(z - \frac{z^3}{6}\right)}{\sqrt{z}} dz. \end{aligned}$$

- ▶ Composite Simpson's rule with  $n = 16$ :

$$\int_0^1 \frac{\sin(z) - \left(z - \frac{z^3}{6}\right)}{\sqrt{z}} dz \approx 0.0014890097.$$

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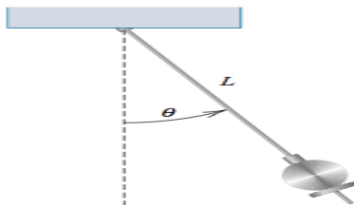
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accurate to 8-digit.

# Initial Value ODE

- ▶ The motion of a swinging pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0,$$



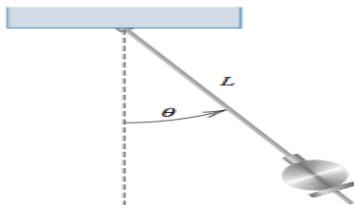
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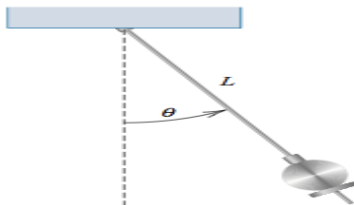
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When does ODE have a solution? How to compute it?

## Lipschitz condition

**Definition:** function  $f(t, y)$  satisfies a **Lipschitz condition** in the variable  $y$  on a set  $D \subset \mathbf{R}^2$  if a constant  $L > 0$  exists with

$$|f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2|,$$

whenever  $(t, y_1), (t, y_2)$  are in  $D$ .  $L$  is Lipschitz constant.

- ▶ **Example 1:** Show that  $f(t, y) = t|y|$  satisfies a Lipschitz condition on the region

$$D = \{(t, y) \mid 0 \leq t \leq T\}.$$

**Solution:** For any  $(t, y_1), (t, y_2)$  in  $D$ ,

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| \leq t |y_1 - y_2| \leq L |y_1 - y_2|,$$

for  $L = T$ .



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whenever  $(t, y_1), (t, y_2)$  are in  $D$ .  $L$  is Lipschitz constant.

- ▶ **Example 2:** Show that  $f(t, y) = t y^2$  does not satisfy any Lipschitz condition on the region

$$D = \{(t, y) \mid 0 \leq t \leq T\}.$$

**Solution:** Choose  $(T, y_1), (T, y_2)$  in  $D$  with  $y_1 = 0, y_2 > 0$ ,

$$\frac{|f(T, y_1) - f(T, y_2)|}{|y_1 - y_2|} = T y_2,$$

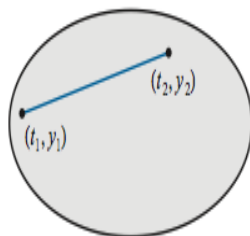
which can be larger than  $L$  for any fixed  $L > 0$ .

# Convex Set

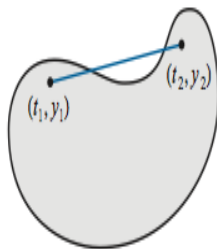
**Definition:** A set  $D \subset \mathbf{R}^2$  is convex if

whenever  $(t_1, y_1)$  and  $(t_2, y_2) \in D$

→ line segment  $(1 - \lambda)(t_1, y_1) + \lambda(t_2, y_2) \in D$  for all  $\lambda \in [0, 1]$ .



Convex



Not convex

**Theorem:** Suppose  $f(t, y)$  is defined on a convex set  $D \subset \mathbf{R}^2$ . If a constant  $L > 0$  exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all } (t, y) \in D,$$

then  $f$  satisfies a Lipschitz condition with Lipschitz constant  $L$ .

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$$D = \{(t, y) \mid 0 \leq t \leq T, \quad -Y \leq y \leq Y\}.$$

**Solution:**

$$\frac{\partial f}{\partial y}(t, y) = 2ty, \quad \left| \frac{\partial f}{\partial y}(t, y) \right| \leq 2TY \quad \text{for all } (t, y) \in D.$$

so  $f(t, y) = t y^2$  satisfies Lipschitz condition with  $L = 2TY$ .

## What is going on with $f(t, y) = t y^2$ ?

- ▶  $f(t, y) = t y^2$  satisfies Lipschitz condition on the region

$$D = \{(t, y) \mid 0 \leq t \leq T, \quad -Y \leq y \leq Y\}.$$

- ▶  $f(t, y) = t y^2$  doesn't satisfy Lipschitz condition on region

$$D = \{(t, y) \mid 0 \leq t \leq T\}.$$

Initial value problem

$$y'(t) = t y^2(t), \quad y(t_0) = \alpha > 0$$

has unique, but unbounded solution

$$y(t) = \frac{2\alpha}{2 + \alpha(t_0^2 - t^2)},$$

the denominator of which vanishes at

$$t = \sqrt{\frac{2}{\alpha} + t_0^2}.$$

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- ▶ for  $|t_0| < T$ , ODE has unique solution on

$$D = \{(t, y) \mid 0 \leq t \leq T, \quad -Y \leq y \leq Y\}.$$

- ▶ for  $\sqrt{\frac{2}{\alpha} + t_0^2} < T$  ODE solution breaks down at  $t = \sqrt{\frac{2}{\alpha} + t_0^2}$   
on

$$D = \{(t, y) \mid 0 \leq t \leq T\}.$$