Inverse matrices

- A square matrix $A \in \mathbf{R}^{n \times n}$ is **non-singular** if a matrix $B \in \mathbf{R}^{n \times n}$ such that

$$A\,B = B\,A = I.$$

  $B \overset{def}{=} A^{-1}$ is **inverse** of A.

- A matrix without an inverse is **singular**.

examples

- $$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad A^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad A \text{ is non-singular.}$$

- $$C = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}, \quad C \text{ is singular.}$$

Solving $A\mathbf{x} = \mathbf{b}$ with Inverse

- If A^{-1} is available,

$$\mathbf{x} = I\,\mathbf{x} = \left(A^{-1}A\right)\,\mathbf{x} = A^{-1}(A\,\mathbf{x}) = A^{-1}\mathbf{b}.$$

- example

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix}, \quad A^{-1} = \frac{1}{9}\begin{pmatrix} -2 & 5 & -1 \\ 4 & -1 & 2 \\ -3 & 3 & 3 \end{pmatrix}, \quad \mathbf{b} \in \mathbf{R}^3,$$

$$\text{solution} \quad \mathbf{x} = \frac{1}{9}\begin{pmatrix} -2 & 5 & -1 \\ 4 & -1 & 2 \\ -3 & 3 & 3 \end{pmatrix}\mathbf{b}.$$

Catch: A^{-1} rarely available, harder to compute than $\mathbf{x}$

Facts about A^{-1}

Assume A^{-1} exists.

- A^{-1} is unique.
- $\left(A^{-1}\right)^{-1} = A$.
- If B^{-1} exists, $A, B \in \mathbf{R}^{n \times n}$, then

$$(A\,B)^{-1} = B^{-1}\,A^{-1}.$$

What if I have to compute A^{-1} anyway?

- A^{-1} is solution to

$$AX = I, \quad X \in \mathbf{R}^{n\times n}.$$

- Partition

$$X = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}, \quad I = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

$$AX = I \leftrightarrow A \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix} \leftrightarrow A\mathbf{x}_j = \mathbf{e}_j,$$

$$\text{for} \quad j = 1, \cdots, n.$$

computing A^{-1} is to solve n systems of equations with same A

Review: GEPP for $A\mathbf{x} = \mathbf{b}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Review: GEPP for $A\mathbf{x} = \mathbf{b}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv} \stackrel{def}{=} \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

Review: GEPP for $A\mathbf{x} = \mathbf{b}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

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  - **eliminating** x_s from E_{s+1} through E_n:

$$\mathbf{a}_{jk} = a_{jk} - \frac{a_{js}}{a_{ss}} a_{sk} \xlongequal{overwrite} a_{jk}, \quad s+1 \le j \le n, \quad s+1 \le k \le n,$$

$$\mathbf{b}_j = b_j - \frac{a_{js}}{a_{ss}} b_s \xlongequal{overwrite} b_j, \quad s+1 \le j \le n.$$

Review: backward substitution after GEPP

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Review: backward substitution after GEPP

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

- for $s = n, n-1, \cdots, 1$:

$$x_s = \frac{b_s - \sum_{k=s+1}^{n} a_{sk} x_k}{a_{ss}}.$$

GEPP for $A\mathbf{X} = \mathbf{B} \in \mathbf{R}^{n \times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

GEPP for $A\mathbf{X} = \mathbf{B} \in \mathbf{R}^{n\times n}$

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- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv} \stackrel{def}{=} \mathbf{argmax}_{s\leq j\leq n}|a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}}.$$

GEPP for $A\mathbf{X} = \mathbf{B} \in \mathbf{R}^{n\times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

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  - **eliminating** x_s from E_{s+1} through E_n:

$$\begin{aligned} \mathbf{a}_{jk} &= a_{jk} - \frac{a_{js}}{a_{ss}}\, a_{sk} \stackrel{overwrite}{=\!=\!=} a_{jk}, \quad s+1 \leq j \leq n, \quad s+1 \leq k \leq n, \\ \mathbf{b}_{jk} &= b_{jk} - \frac{a_{js}}{a_{ss}}\, b_{sk} \stackrel{overwrite}{=\!=\!=} b_{jk}, \quad s+1 \leq j \leq n, \quad 1 \leq k \leq n. \end{aligned}$$

total cost: $\frac{5}{3}\, n^3$ operations

Backward substitution

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

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- for $s = n, n-1, \cdots, 1$:

$$x_{sj} = \frac{b_{sj} - \sum_{k=s+1}^{n} a_{sk} x_{kj}}{a_{ss}}, \quad j = 1, \cdots n.$$

- total cost for backward substitution: n^3 operations.
- total cost for computing A^{-1}: $\frac{8}{3} n^3$ operations.
- computing A^{-1} is 4 times as expensive as solving $A\mathbf{x} = \mathbf{b}$.

Matrix transpose: A^T (I)

$$
\begin{aligned}
A &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbf{R}^{m\times n}, \\
A^T &= \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} \in \mathbf{R}^{n\times m}.
\end{aligned}
$$

Matrix transpose: A^T (I)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbf{R}^{m\times n},$$

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} \in \mathbf{R}^{n\times m}.$$

example

$$A = \begin{pmatrix} 2 & 4 & 7 & 1 \\ 2 & -9 & -1 & 2 \end{pmatrix} \in \mathbf{R}^{2\times 4}, \quad A^T = \begin{pmatrix} 2 & 2 \\ 4 & -9 \\ 7 & -1 \\ 1 & 2 \end{pmatrix} \in \mathbf{R}^{4\times 2}.$$

Matrix transpose: A^T (II)

Theorem:

$$\left(A^T\right)^T = A, \quad (A\,B)^T = B^T\,A^T.$$

If A^{-1} exists, then

$$(A^{-1})^T = \left(A^T\right)^{-1} \quad \square$$

The Determinant of a Matrix

- $$\text{for} \quad A = a, \quad \mathbf{det}\,(A) \stackrel{def}{=} a.$$

- $$\text{for} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{det}\,(A) \stackrel{def}{=} a_{11}\,a_{22} - a_{12}\,a_{21}.$$

- **definition**: **minor** M_{ij} is determinant of $(n-1) \times (n-1)$ submatrix of A without row i and column j.

$$\mathbf{det}\,(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij}\, M_{ij}, \quad 1 \leq i \leq n.$$

The Determinant of a Matrix: example

$$B = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix}.$$

- $$\mathbf{det}\,(B) = 2 * (-1) - 2 * 1 = -4.$$

- $$\begin{aligned} \mathbf{det}\,(A) &= (-1)^{1+1} * 1 * \mathbf{det} \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} + (-1)^{1+2} * 2 * \mathbf{det} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \\ &\quad +(-1)^{1+3} * 1 * \mathbf{det} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \\ &= (2 * (-1) - 2 * 1) - 2 * (1 * (-1) - 2 * 1) + (1 * 1 - 2 * 1) \\ &= -4 + 6 - 1 = 1. \end{aligned}$$

How to calculate **det** quickly? What does it do?

Properties of $\mathbf{det}\,(A)$, $\quad A \in \mathbf{R}^{n\times n}$ (I)

- If $\widetilde{A}$ is obtained from A with elementary operation $(E_i) \leftrightarrow (E_j)$ for $i \neq j$, then $\mathbf{det}\left(\widetilde{A}\right) = -\mathbf{det}\,(A)$.
  (ROW SWAPS CHANGE SIGN IN $\mathbf{det}\,(A)$.)
- If $\widetilde{A}$ is obtained from A with elementary operation $(E_i + \lambda\, E_j) \to (E_i)$ for $i \neq j$, then $\mathbf{det}\left(\widetilde{A}\right) = \mathbf{det}\,(A)$.
  (ELIMINATION DOES NOT CHANGE $\mathbf{det}\,(A)$.)
- If A is upper-triangular, then

$$\mathbf{det}\,(A) = \prod_{j=1}^{n} a_{jj}$$

GEPP can be used to compute $\mathbf{det}\,(A)$

Properties of $\mathbf{det}\,(A)$, $\quad A \in \mathbf{R}^{n\times n}$ (II)

- $\mathbf{det}\left(A^T\right) = \mathbf{det}\,(A)$.
- If A has two identical rows, then $\mathbf{det}\,(A) = 0$.
- If $B \in \mathbf{R}^{n\times n}$, then $\mathbf{det}\,(A\,B) = \mathbf{det}\,(A)\;\mathbf{det}\,(B)$.
- If A^{-1} exists, then $\mathbf{det}\left(A^{-1}\right) = (\mathbf{det}\,(A))^{-1}$.

Properties of **det** (A), $A \in \mathbf{R}^{n \times n}$ (III)

$$
\begin{aligned}
& \mathbf{det}\,(A) \neq 0 \\
\leftrightarrow \quad & A^{-1} \quad \text{exists} \\
\leftrightarrow \quad & A\,\mathbf{x} = \mathbf{b} \quad \text{has a unique solution for any } b \in \mathbf{R}^n. \\
\leftrightarrow \quad & \text{GEPP can be performed on } A\,\mathbf{x} = \mathbf{b} \text{ for any } b \in \mathbf{R}^n.
\end{aligned}
$$

GEPP works for $\mathbf{det}\,(A)$ (I)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

- **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv} \stackrel{def}{=} \mathbf{argmax}_{1 \le j \le n} |a_{j1}|, \quad (|a_{\mathbf{piv},1}| = \mathbf{max}_{1 \le j \le n} |a_{j1}|)$$

- exchange rows E_1 and $E_{\mathbf{piv}}$ $\quad (E_1 \leftrightarrow E_{\mathbf{piv}}.)$
  ($\mathbf{det}\,(A)$ CHANGES SIGN IF $\mathbf{piv} \neq 1$; $\mathbf{det}\,(A) = 0$ IF $a_{11} = 0$.)
- elimination of x_1 from E_2 through E_n:

$$\left(E_j - \frac{a_{j1}}{a_{11}} E_1\right) \to (E_j)\,, \quad 2 \le j \le n.$$

($\mathbf{det}\,(A)$ DOES NOT CHANGE)

GEPP works for $\mathbf{det}\,(A)$ (II)

$$\textbf{new} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \overset{overwrite}{=\!=\!=} A,$$

$$\text{where} \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}}\, a_{1k}, \quad 2 \le j \le n, \quad 2 \le k \le n.$$

- repeat same process til A becomes upper triangular

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

- determinant of the original matrix A is

$$(-1)^{\#\text{ of row swaps}} \prod_{1 \le j = \le n} a_{jj}.$$

General linear equations

$$
\begin{array}{llllllllll}
E_1 : & a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\
E_2 : & a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\
\vdots & & \vdots & & & & & \vdots & \\
E_n : & a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n,
\end{array}
$$

General linear equations

$$
\begin{array}{lcccccccc}
E_1: & a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\
E_2: & a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\
\vdots & & \vdots & & & & & \vdots & \\
E_n: & a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n,
\end{array}
$$

$$
A \stackrel{def}{=} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} \stackrel{def}{=} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} \stackrel{def}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
$$

General linear equations

$$
\begin{array}{rcccccccc}
E_1 : & a_{11}x_1 & + & a_{12}x_2 & \cdots & + & a_{1n}x_n & = & b_1, \\
E_2 : & a_{21}x_1 & + & a_{22}x_2 & \cdots & + & a_{2n}x_n & = & b_2, \\
\vdots & & \vdots & & & & & \vdots & \\
E_n : & a_{n1}x_1 & + & a_{n2}x_2 & \cdots & + & a_{nn}x_n & = & b_n,
\end{array}
$$

$$
A \stackrel{def}{=} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{b} \stackrel{def}{=} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad \mathbf{x} \stackrel{def}{=} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
$$

- equation E_j maps to **row** j of A: $\left(a_{j1}, a_{j2}, \cdots, a_{jn}\right)$, $1 \leq j \leq n$.
- variable x_k relates to **column** k of A: $\begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix}$, $1 \leq k \leq n$.
- will concentrate on A for now.

Gifts from Math God in GE (I), assuming $a_{11} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \text{first column} \ = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

- elimination in rows E_2 through E_n, on first column:

$$l_{j1} \stackrel{def}{=} \frac{a_{j1}}{a_{11}}, \quad (E_j - l_{j1} E_1) \to (E_j), \quad 2 \leq j \leq n.$$

$$\textbf{new} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

$$\text{where} \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k}, \quad 2 \leq j \leq n, \quad 2 \leq k \leq n.$$

Gifts from Math God in GE (II), assuming $a_{11} \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \textbf{new} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

$$\text{where} \quad \mathbf{a}_{jk} = a_{jk} - l_{j1}\, a_{1k}, \quad l_{j1} = \frac{a_{j1}}{a_{11}}, \quad 2 \le j \le n, \quad 2 \le k \le n.$$

- *Gift* #1: Matrix-matrix product connection

$$A = L_1 \cdot (\textbf{new} \quad A), \quad \text{where} \quad L_1 = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix}.$$

Gifts from Math God in GE (III), assuming $a_{11} \neq 0$

$$A = L_1 \cdot (\textbf{new} \quad A) = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

Gifts from Math God in GE (III), assuming $a_{11} \neq 0$

$$A = L_1 \cdot (\textbf{new} \quad A) = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

- new step of elimination, assuming $\mathbf{a}_{22} \neq 0$,

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} 1 & & & \\ \mathbf{l}_{32} & 1 & & \\ \vdots & & \ddots & \\ \mathbf{l}_{n2} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} & \cdots & \mathbf{a}_{2n} \\ 0 & \widehat{a}_{33} & \cdots & \widehat{a}_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widehat{a}_{n3} & \cdots & \widehat{a}_{nn} \end{pmatrix}$$

Gifts from Math God in GE (III), assuming $a_{11} \neq 0$

$$A = L_1 \cdot (\textbf{new} \quad A) = \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},$$

- new step of elimination, assuming $\mathbf{a}_{22} \neq 0$,

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} 1 & & & \\ \mathbf{l}_{32} & 1 & & \\ \vdots & & \ddots & \\ \mathbf{l}_{n2} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_{22} & \mathbf{a}_{23} & \cdots & \mathbf{a}_{2n} \\ 0 & \widehat{a}_{33} & \cdots & \widehat{a}_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \widehat{a}_{n3} & \cdots & \widehat{a}_{nn} \end{pmatrix}$$

- *Gift* #2: "free" matrix-matrix product

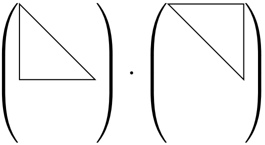
$$A = \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & \mathbf{l}_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ l_{n1} & \mathbf{l}_{n2} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \mathbf{a}_{23} & \cdots & \mathbf{a}_{2n} \\ 0 & 0 & \widehat{a}_{33} & \cdots & \widehat{a}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \widehat{a}_{n3} & \cdots & \widehat{a}_{nn} \end{pmatrix}.$$

Gifts from Math God in GE (IV), GE=LU factorization

- GE for $n = 2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ 0 & \mathbf{a}_{22} \end{pmatrix}$$

$$\stackrel{def}{=} L \cdot U =$$



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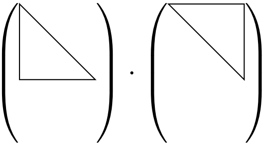


,

- GE for $n \geq 3$:

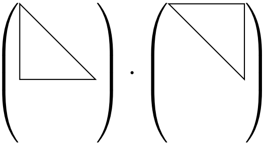
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \left(\mathbf{l} \stackrel{def}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix} \right).$$

Gifts from Math God in GE (V), GE=LU factorization

- Induction hypothesis:

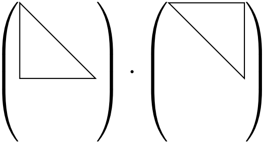
$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{L} \cdot \mathbf{U} =$$



Gifts from Math God in GE (V), GE=LU factorization

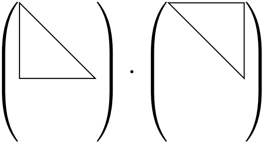
- Induction hypothesis:

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{L} \cdot \mathbf{U} =$$



- *Gift* #3: LU factorization

$$\begin{aligned} A &= \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ \mathbf{0} & \mathbf{L}\,\mathbf{U} \end{pmatrix}, \\ &= \begin{pmatrix} 1 & \\ \mathbf{l} & \mathbf{L} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ \mathbf{0} & \mathbf{U} \end{pmatrix} \\ &\overset{def}{=} L \cdot U = \end{aligned}$$



Gaussian Elimination as LU factorization

- In Gaussian Elimination: for $s = 1, \cdots, n-1$,

$$
\begin{aligned}
l_{js} &= \frac{a_{js}}{a_{ss}}, \quad 1+s \leq j \leq n, \\
a_{jk} &\overset{overwrite}{=\!=\!=} a_{jk} - l_{js}\, a_{sk}, \quad 1+s \leq j,\, k \leq n.
\end{aligned}
$$

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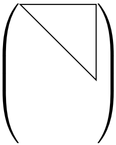
\end{aligned}
$$

- A becomes upper-triangular after GE:

$$A \xrightarrow{GE}$$



$$\overset{def}{=} U \overset{\text{notation}}{=\!=} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{pmatrix}.$$

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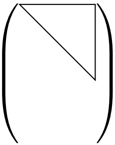
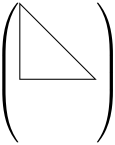
$A \xrightarrow{GE}$ 

$$
\overset{def}{=} U \overset{\text{notation}}{=\!=} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{pmatrix}.
$$

- LU factorization

$$
A = L \cdot U, \qquad \text{where } L \overset{def}{=} \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} =
$$

 .

Gaussian Elimination as LU factorization, example

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix} \in \mathbf{R}^{4\times 4}$$

$$\begin{aligned} A &= \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ 3 & & 1 & \\ -1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ & -1 & -1 & -5 \\ & -4 & -1 & -7 \\ & 3 & 3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & \\ 2 & 1 & & \\ 3 & 4 & 1 & \\ -1 & -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 3 \\ & -1 & -1 & -5 \\ & & 3 & 13 \\ & & & -13 \end{pmatrix}. \end{aligned}$$

Row interchange vs. Permutation Matrix

- **definition**: A permutation matrix $P = (p_{ij})$ is a matrix obtained by rearranging the rows of the identity matrix I_n.

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- 3×3 **example**

$$P_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

  - $P_{2,3} \cdot A$ is A with interchanged rows:

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  - Let $P_{k,s}$ be permutation interchanging rows k and s of I_n.
    For any $A = (a_{ij}) \in \mathbf{R}^{n \times n}$,
    $P_{k,s} \cdot A$ is *A with rows k and s interchanged*.

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  - Let $P_{k,s}$ be permutation interchanging rows k and s of I_n.
    For any $A = (a_{ij}) \in \mathbf{R}^{n \times n}$,
    $P_{k,s} \cdot A$ is $\underline{A \text{ with rows } k \text{ and } s \text{ interchanged}}$.
- Let P be a permutation, then $P\,P_{k,s}$ is also a permutation.

Gaussian Elimination with partial pivoting: review

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

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- for $s = 1, 2, \cdots, n-1$:
  - **pivoting**: choose largest entry in absolute value:

$$\mathbf{piv}_s \overset{def}{=} \mathbf{argmax}_{s \le j \le n} |a_{js}|, \quad E_s \leftrightarrow E_{\mathbf{piv}_s}$$

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  - **eliminating** x_s from E_{s+1} through E_n:

$$\begin{aligned} l_{js} &= \frac{a_{js}}{a_{ss}}, \quad s+1 \le j \le n, \\ a_{jk} &= a_{jk} - l_{js}\, a_{sk}, \quad s+1 \le j,\, k \le n. \end{aligned}$$

GEPP as LU factorization, example

$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \in \mathbf{R}^{4\times 4}$$

$$P_{1,2} \cdot A = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ & 0 & -1 & 1 \\ & 0 & 1 & 2 \\ & 1 & 1 & 0 \end{pmatrix}.$$

$$P_{1,2} \cdot A = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ & 0 & -1 & 1 \\ & 0 & 1 & 2 \\ & 1 & 1 & 0 \end{pmatrix}.$$

$$\begin{aligned} P_{1,3} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 2 \\ & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 2 \\ & & 3 \end{pmatrix}. \end{aligned}$$

$$P_{1,2} \cdot A = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ & 0 & -1 & 1 \\ & 0 & 1 & 2 \\ & 1 & 1 & 0 \end{pmatrix},$$

$$P_{1,3} \cdot \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 2 \\ & & 3 \end{pmatrix}.$$

- Permutation

$$P = \begin{pmatrix} 1 & \\ & P_{1,3} \end{pmatrix} \cdot P_{1,2} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix} \cdot \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \\ 1 & & & \end{pmatrix}.$$

$$P_{1,2}\cdot A = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ -1 & & 1 & \\ 1 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ & 0 & -1 & 1 \\ & 0 & 1 & 2 \\ & 1 & 1 & 0 \end{pmatrix},$$

$$P_{1,3}\cdot \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 2 \\ & & 3 \end{pmatrix}.$$

- LU factorization

$$P\cdot A = \begin{pmatrix} 1 & & \\ & P_{1,3}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} & P_{1,3} \end{pmatrix} \cdot \begin{pmatrix} 1 & \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \\ & \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \\ & P_{1,3}\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \end{pmatrix}.$$

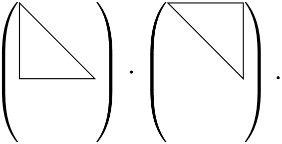
$$A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} & 1 & & \\ & & & 1 \\ & & 1 & \\ 1 & & & \end{pmatrix}.$$

$$\begin{aligned} P \cdot A &= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & & 1 & \\ 0 & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \\ & \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 2 \\ & & 3 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 & 2 \\ & 1 & 1 & 0 \\ & & 1 & 2 \\ & & & 3 \end{pmatrix} \overset{def}{=} L \cdot U. \end{aligned}$$

GEPP as LU factorization

Theorem: Let $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ be non-singular. Then GEPP computes an LU factorization with permutation matrix P such that

$$P \cdot A = L \cdot U =$$



$P \cdot A = L \cdot U$, Proof by Induction

- GEPP for $n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.
  - **pivoting**:

$$\textbf{piv}_1 \stackrel{def}{=} \textbf{argmax}_{1 \leq j \leq 2} |a_{j1}|, \quad P = \begin{cases} I, & \text{if } \textbf{piv}_1 = 1, \\ P_{1,2} & \text{if } \textbf{piv}_1 = 2. \end{cases}$$

  - **elimination**:

$$P \cdot A = L \cdot U.$$

$P \cdot A = L \cdot U$, Proof by Induction

- GEPP for $n \geq 3$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$:
  - **pivoting**:

$$\mathbf{piv}_1 \stackrel{def}{=} \mathbf{argmax}_{1 \leq j \leq n} |a_{j1}|, \quad \overline{P} = \begin{cases} I, & \text{if } \mathbf{piv}_1 = 1, \\ P_{1,\mathbf{piv}_1} & \text{if } \mathbf{piv}_1 \geq 2. \end{cases}$$

  - **elimination**:

$$\overline{P} \cdot A = \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \qquad \left(\mathbf{l} \stackrel{def}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix} . \right)$$

- Induction hypothesis:

$$\mathbf{P} \cdot \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{L} \cdot \mathbf{U}.$$

- Induction hypothesis:

$$\mathbf{P} \cdot \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{L} \cdot \mathbf{U}.$$

- putting it together,

$$\begin{pmatrix} 1 & \\ & \mathbf{P} \end{pmatrix} \cdot \overline{P} \cdot A = \begin{pmatrix} 1 & \\ & \mathbf{P} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ & \vdots & \ddots & \vdots \\ & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}$$

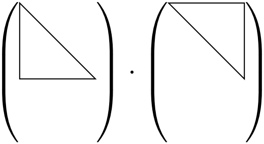
$$= \begin{pmatrix} 1 & \\ \mathbf{P} \cdot \mathbf{l} & \mathbf{P} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ & \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \\ \mathbf{P} \cdot \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ & \mathbf{P} \cdot \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \end{pmatrix}$$

$$\text{Let} \quad P \stackrel{def}{=} \begin{pmatrix} 1 & \\ & \mathbf{P} \end{pmatrix} \cdot \overline{P}.$$

- LU magic:

$$\begin{aligned} P \cdot A \quad &= \quad \begin{pmatrix} 1 & \\ \mathbf{P} \cdot \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ & \mathbf{L} \cdot \mathbf{U} \end{pmatrix} \\ &= \quad \begin{pmatrix} 1 & \\ \mathbf{P} \cdot \mathbf{l} & \mathbf{L} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \begin{pmatrix} a_{12} & \cdots & a_{1n} \end{pmatrix} \\ & \mathbf{U} \end{pmatrix} \end{aligned}$$

$\stackrel{def}{=} \quad L \cdot U =$  .

Solving general linear equations with GEPP

$$A\mathbf{x} = \mathbf{b}, \quad P \cdot A = L \cdot U$$

- interchanging components in $\mathbf{b}$

$$P \cdot (A\mathbf{x}) = (P \cdot \mathbf{b}), \quad (L \cdot U)\,\mathbf{x} = (P \cdot \mathbf{b}).$$

- solving for $\mathbf{b}$ with forward and backward substitution

$$\begin{aligned} \mathbf{x} &= (L \cdot U)^{-1} (P \cdot \mathbf{b}) \\ &= \left(U^{-1} \left(L^{-1} (P \cdot \mathbf{b})\right)\right). \end{aligned}$$

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Cost Analysis

- computing $P \cdot A = L \cdot U$: about $2/3n^3$ operations.
- forward and backward substitution: about $2n^2$ operations.
- most important to compute $P \cdot A = L \cdot U$ **efficiently**

Strictly Diagonally Dominant (**SDD**) Matrices

- **Definition**: Matrix $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ is **SDD** if

$$|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}| \quad \text{holds for each} \quad i = 1, 2, \cdots, n.$$

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- Example I: matrix $A = \begin{pmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{pmatrix}$ is **SDD**.

$$|7| > |2| + |0|, \; |5| > |3| + |-1|, \; |-6| > |0| + |5|.$$

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- Example II: matrix $B = \begin{pmatrix} 7 & 5 & 0 \\ 3 & 5 & -1 \\ 0 & -3 & 3 \end{pmatrix}$ is NOT **SDD**.

$$|3| \leq |0| + |-3|.$$

GE on **SDD**: succeeds without pivoting (I)

- Let $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ be **SDD**, so

$$|a_{11}| > \sum_{j=1, j\neq 1}^{n} |a_{1j}| \geq 0.$$

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- elimination with $a_{11} \neq 0$:

$$\begin{array}{rcl} A & = & L_1 \cdot (\textbf{new} \quad A) \\ & = & \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \end{array}$$

$$l_{j1} = \frac{a_{j1}}{a_{11}}, \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k}, \quad 2 \leq j \leq n, \quad 2 \leq k \leq n.$$

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- **only do**: show $\mathbf{A} \stackrel{def}{=} \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}$ remains **SDD**.

GE on **SDD**: only need to show **A** remains **SDD**

$$|a_{11}| \quad > \quad \sum_{j=1, j\neq 1}^{n} |a_{1j}|, \quad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 2, \cdots n,$$

$$\mathbf{A} \quad = \quad \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \mathbf{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}.$$

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- for $i = 2, \cdots n$,

$$\sum_{j=\mathbf{2}, j\neq i}^{n} |\mathbf{a}_{ij}| = \sum_{j=\mathbf{2}, j\neq i}^{n} \left| a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right| \leq \left(\sum_{j=\mathbf{2}, j\neq i}^{n} |a_{ij}| \right) + \left| \frac{a_{i1}}{a_{11}} \right| \left(\sum_{j=\mathbf{2}, j\neq i}^{n} |a_{1j}| \right)$$

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$$|a_{11}| \; > \; \sum_{j=1, j\neq 1}^{n} |a_{1j}|\,, \quad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|\,, \quad i = 2, \cdots n,$$

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- for $i = 2, \cdots n,$

$$\sum_{j=\mathbf{2}, j\neq i}^{n} |\mathbf{a}_{ij}| = \sum_{j=\mathbf{2}, j\neq i}^{n} \left| a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j} \right| \leq \left(\sum_{j=\mathbf{2}, j\neq i}^{n} |a_{ij}| \right) + \left| \frac{a_{i1}}{a_{11}} \right| \left(\sum_{j=\mathbf{2}, j\neq i}^{n} |a_{1j}| \right)$$

$$\stackrel{\mathbf{SDD}}{<} \; (|a_{ii}| - |a_{i1}|) + \left| \frac{a_{i1}}{a_{11}} \right| (|a_{11}| - |a_{1i}|) = |a_{ii}| - \left| \frac{a_{i1}}{a_{11}} \right| |a_{1i}|$$

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$$|a_{11}| \quad > \quad \sum_{j=1, j\neq 1}^{n} |a_{1j}|, \quad |a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 2, \cdots n,$$

$$\mathbf{A} \quad = \quad \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \mathbf{a}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}.$$

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$$\leq \quad \left| a_{ii} - \frac{a_{i1}}{a_{11}} a_{1i} \right| = |\mathbf{a}_{ii}|.$$

GE on **SDD**: example

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{pmatrix} \quad \text{is } \mathbf{SDD}.$$

$$A = \begin{pmatrix} 1 & & \\ \frac{3}{7} & 1 & \\ 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 0 \\ & \frac{29}{7} & -1 \\ & 5 & -6 \end{pmatrix} \quad \left[\begin{pmatrix} \frac{29}{7} & -1 \\ 5 & -6 \end{pmatrix} \text{ is } \mathbf{SDD} \right]$$

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$$A = \begin{pmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{pmatrix} \quad \text{is } \textbf{SDD}.$$

$$\begin{aligned} A &= \begin{pmatrix} 1 & & \\ \frac{3}{7} & 1 & \\ 0 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 0 \\ & \frac{29}{7} & -1 \\ & 5 & -6 \end{pmatrix} \quad \left[\begin{pmatrix} \frac{29}{7} & -1 \\ 5 & -6 \end{pmatrix} \text{ is } \textbf{SDD} \right] \\ &= \begin{pmatrix} 1 & & \\ \frac{3}{7} & 1 & \\ 0 & \frac{35}{29} & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 0 \\ & \frac{29}{7} & -1 \\ & & -\frac{139}{29} \end{pmatrix} \end{aligned}$$

Symmetric Positive Definite Matrices

- **Definition**: Matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is **SPD** if

$$A = A^T, \quad \mathbf{x}^T A\mathbf{x} > 0 \quad \text{for any non-zero} \quad \mathbf{x} \in \mathbf{R}^n.$$

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- Example I: matrix $A = \begin{pmatrix} \mathbf{0} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = A^T$ is NOT **SPD**.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \begin{pmatrix} \mathbf{0} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

Definition: $A = A^T$ is **SPD** if $\mathbf{x}^T A \mathbf{x} > 0$ for any $\mathbf{x} \neq 0$

- Example II: matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = A^T$ is **SPD**.
  - Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$.

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  - Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3$.

$$\begin{aligned}
\mathbf{x}^T A \mathbf{x} &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix} \\
&= 2x_1^2 - 2x_1 x_2 + 2x_2^2 - 2x_2 x_3 + 2x_3^2
\end{aligned}$$

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&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{pmatrix} \\
&= 2x_1^2 - 2x_1 x_2 + 2x_2^2 - 2x_2 x_3 + 2x_3^2 \\
&\overset{\textit{rearranging}}{=\!=\!=} x_1^2 + \left(x_1^2 - 2x_1 x_2 + x_2^2\right) + \left(x_2^2 - 2x_2 x_3 + x_3^2\right) + x_3^2 \\
&= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0, \quad \text{unless } \mathbf{x} = 0.
\end{aligned}
$$

GE on **SPD**: succeeds without pivoting (I)

- Let $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ be **SPD**, then

$$
\begin{aligned}
A^T &= A, \quad \text{therefore} \quad a_{jk} = a_{kj} \quad \text{for all } 1 \le j, k \le n \\
a_{11} &= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot A \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} > 0.
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- elimination without pivoting

$$
\begin{aligned}
A &= L_1 \cdot (\textbf{new} \quad A) \\
&= \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & & \ddots & \\ l_{n1} & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix},
\end{aligned}
$$

$$
l_{j1} = \frac{a_{j1}}{a_{11}}, \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k}, \quad 2 \le j \le n, \quad 2 \le k \le n.
$$

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- symmetry in A implies for $2 \leq j, k \leq n$,

$$l_{j1} = \frac{a_{j1}}{a_{11}} = \frac{a_{1j}}{a_{11}}, \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k} = \mathbf{a}_{kj}.$$

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- $$\mathbf{l}_1 \stackrel{def}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix}, \quad \mathbf{A} \stackrel{def}{=} \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \text{and}$$

$$\begin{aligned} A &= \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{11}\mathbf{l}_1^T \\ & \mathbf{A} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{l}_1^T \\ & I_{n-1} \end{pmatrix} \end{aligned}$$

- symmetric matrix factorization

$$A = \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{l}_1^T \\ & I_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix}^T,$$

$$\mathbf{A} \stackrel{def}{=} \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix}, \quad \mathbf{a}_{jk} = a_{jk} - \frac{a_{j1}}{a_{11}} a_{1k} = \mathbf{a}_{kj},$$

Only need to compute $\mathbf{a}_{jk}$ for $j \leq k$ due to symmetry

- **next step**: show $\mathbf{A}$ remains **SPD**.

$$A = \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix}^T,$$

- Let $\widehat{\mathbf{x}} \in \mathbf{R}^{n-1}$ be any non-zero vector.
  Must show $\widehat{\mathbf{x}}^T \cdot \mathbf{A}\widehat{\mathbf{x}} > 0$ for $\mathbf{A}$ to be **SPD**.
-

Note $\mathbf{x} \stackrel{def}{=} \begin{pmatrix} -\widehat{\mathbf{x}}^T \mathbf{l}_1 \\ \widehat{\mathbf{x}} \end{pmatrix} \in \mathbf{R}^n$ is non-zero, therefore $\mathbf{x}^T \cdot A \cdot \mathbf{x} > 0.$

$$A = \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix}^T,$$

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- consequently,

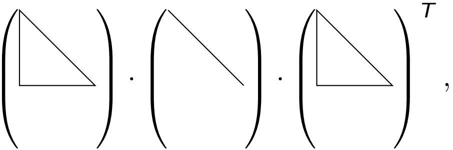
$$\begin{aligned} 0 &< \mathbf{x}^T \cdot A \cdot \mathbf{x} \\ &= \begin{pmatrix} -\mathbf{l}_1^T \widehat{\mathbf{x}} \\ \widehat{\mathbf{x}} \end{pmatrix}^T \cdot \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{A} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathbf{l}_1 & I_{n-1} \end{pmatrix}^T \cdot \begin{pmatrix} -\mathbf{l}_1^T \widehat{\mathbf{x}} \\ \widehat{\mathbf{x}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \widehat{\mathbf{x}}^T \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{A} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \widehat{\mathbf{x}} \end{pmatrix} \\ &= \widehat{\mathbf{x}}^T \cdot \mathbf{A} \cdot \widehat{\mathbf{x}}. \end{aligned}$$

Cholesky factorization for SPD matrix: $A = L\,D\,L^T$

- Cholesky for $n = 2$:

$$
\begin{aligned}
A &= \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix} \quad [A \text{ is symmetric}] \\
&= \begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{a}_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix}^T, \quad \left[l_{21} = \frac{a_{21}}{a_{11}} \right]
\end{aligned}
$$

$$\overset{def}{=} L \cdot D \cdot L^T =$$



Cholesky factorization for SPD matrix: $A = L\,D\,L^T$

- Cholesky for $n \geq 3$:

$$A = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{[A is symmetric]}$$

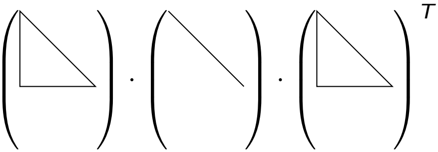
$$= \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & & & \\ & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ & \vdots & \ddots & \vdots \\ & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix}^T,$$

where $\mathbf{l} \stackrel{def}{=} \begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix}.$

Cholesky factorization for SPD matrix: $A = L\,D\,L^T$

- induction hypothesis

$$\begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} = \mathbf{LDL}^T =$$



Cholesky factorization for SPD matrix: $A = L\,D\,L^T$

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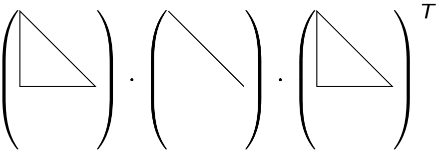
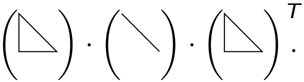


- Cholesky factorization

$$A = \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \begin{pmatrix} \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix}^T,$$

$$= \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{LDL}^T \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & I_{n-1} \end{pmatrix}^T,$$

$$= \begin{pmatrix} 1 & \\ \mathbf{l} & \mathbf{L} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \\ & \mathbf{D} \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \mathbf{l} & \mathbf{L} \end{pmatrix}^T =$$



Cholesky factorization $A = L\,D\,L^T$: example

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{pmatrix} = A^T.$$

$$A \quad = \quad \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & & \\ & 4 & 3 \\ & 3 & \frac{13}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix}^T$$

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$$\begin{aligned}
A &= \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & & \\ & 4 & 3 \\ & 3 & \frac{13}{4} \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & \\ & \begin{pmatrix} 1 & \\ \frac{3}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \frac{3}{4} & 1 \end{pmatrix}^T \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix}^T
\end{aligned}$$

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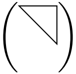
&= \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & \\ & \begin{pmatrix} 1 & \\ \frac{3}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \\ \frac{3}{4} & 1 \end{pmatrix}^T \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & & 1 \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{3}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & & \\ & 4 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ \frac{1}{4} & \frac{3}{4} & 1 \end{pmatrix}^T.
\end{aligned}$$

Cholesky factorization is a special LU factorization

- $A = L\,D\,L^T = L\,U$, with $U = D\,L^T =$ 
- only L need to be computed, saving half of the work in factorization.
- total cost: about $\frac{1}{3}n^3$ operations.

Review on Natural Splines: equations in matrix form

Equations for spline coefficients $\{c_j\}_{j=1}^{n-1}$,

$$A \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix} = \mathbf{rhs},$$

where

$$A \stackrel{def}{=} \begin{pmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{pmatrix}.$$

A is **SDD**, **SPD**, and **tri-diagonal**

Special matrix: **tri-diagonal** matrix

$A \in \mathbf{R}^{n\times n}$ is **tri-diagonal** if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

LU should simplify since A has so many zero entries

tri-diagonal LU factorization

Let $l_{21} = \frac{a_{21}}{a_{11}}$ and $\mathbf{a}_{22} = a_{22} - l_{21}\,a_{12}$, then

$$A \;=\; \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & & & \\ & \mathbf{a}_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix}.$$

elimination only means computing l_{21} and $\mathbf{a}_{22}$

tri-diagonal LU factorization

Let $\mathbf{a}_{11} = a_{11}$, then

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

tri-diagonal LU factorization

Let $\mathbf{a}_{11} = a_{11}$, then

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ & \ddots & \ddots & & \\ & & l_{n-1,n-2} & 1 & \\ & & & l_{n,n-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a}_{11} & a_{12} & & & \\ & \mathbf{a}_{22} & a_{23} & & \\ & & \ddots & \ddots & \\ & & & \mathbf{a}_{n-1,n-1} & a_{n-1,n} \\ & & & & \mathbf{a}_{n,n} \end{pmatrix}$$

$$l_{j+1,j} = \frac{a_{j+1,j}}{\mathbf{a}_{jj}}, \quad \mathbf{a}_{j+1,j+1} = a_{j+1,j+1} - l_{j+1,j}\, a_{j,j+1}, \quad j = 1, \cdots, n-1.$$

grand total: $3n$ operations. Pivoting at $O(n)$ additional operations