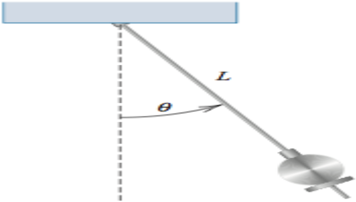
Initial Value ODE

- The motion of a swinging pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0,$$



- Initial Value conditions

$$\theta(t_0) = \theta_0, \quad \text{and} \quad \theta'(t_0) = \theta'_0.$$

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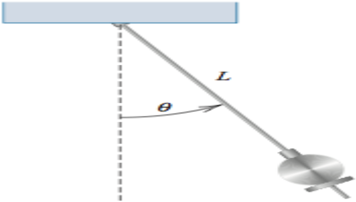

- Initial Value conditions

$$\theta(t_0) = \theta_0, \quad \text{and} \quad \theta'(t_0) = \theta_0'.$$

When does ODE have a solution? How to compute it?

Lipschitz condition

Definition: function $f(t, y)$ satisfies a **Lipschitz condition** in the variable y on a set $D \subset \mathbf{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L\,|y_1 - y_2|\,,$$

whenever $(t, y_1), (t, y_2)$ are in D. L is Lipschitz constant.

- **Example 1**: Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the region

  $$D = \{(t, y) \quad | \quad 0 \leq t \leq T\}\,.$$

  Solution: For any $(t, y_1), (t, y_2)$ in D,

  $$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| \leq t\,|y_1 - y_2| \leq L\,|y_1 - y_2|\,,$$

  for $L = T$.

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$$|f(t,y_1) - f(t,y_2)| \leq L\,|y_1 - y_2|\,,$$

whenever $(t,y_1),(t,y_2)$ are in D. L is Lipschitz constant.

- **Example 2**: Show that $f(t,y) = t\,y^2$ does not satisfy any Lipschitz condition on the region

$$D = \{(t,y) \quad | \quad 0 \leq t \leq T\}\,.$$

Solution: Choose $(T,y_1),(T,y_2)$ in D with $y_1 = 0, y_2 > 0$,

$$\frac{|f(T,y_1) - f(T,y_2)|}{|y_1 - y_2|} = T\,y_2,$$

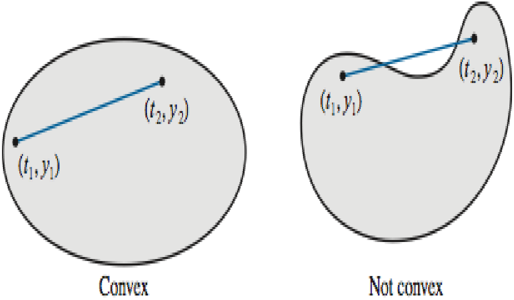
which can be larger than L for any fixed $L > 0$.

Convex Set

Definition: A set $D \subset \mathbf{R}^2$ is convex if

whenever (t_1, y_1) and $(t_2, y_2) \in D$

$\longrightarrow$ line segment $(1-\lambda)(t_1, y_1) + \lambda(t_2, y_2) \in D$ for all $\lambda \in [0, 1]$.



Theorem: Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbf{R}^2$. If a constant $L > 0$ exists with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L, \quad \text{for all} \quad (t, y) \in D,$$

then f satisfies a Lipschitz condition with Lipschitz constant L.

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- **Example 1**: Show that $f(t, y) = t\,y^2$ satisfies Lipschitz condition on the region

$$D = \{(t, y) \quad | \quad 0 \leq t \leq T, \quad -Y \leq y \leq Y\}.$$

Solution:

$$\frac{\partial f}{\partial y}(t, y) = 2ty, \quad \left|\frac{\partial f}{\partial y}(t, y)\right| \leq 2T\,Y \quad \text{for all} \quad (t, y) \in D.$$

so $f(t, y) = t\,y^2$ satisfies Lipschitz condition with $L = 2T\,Y$.

What is going on with $f(t,y) = t\,y^2$?

- $f(t,y) = t\,y^2$ satisfies Lipschitz condition on the region

$$D = \{(t,y) \quad | \quad 0 \le t \le T, \quad -Y \le y \le Y\}.$$

- $f(t,y) = t\,y^2$ doesn't satisfy Lipschitz condition on region

$$D = \{(t,y) \quad | \quad 0 \le t \le T\}.$$

Initial value problem

$$y'(t) = t\,y^2(t), \quad y(t_0) = \alpha > 0$$

has unique, but unbounded solution

$$y(t) = \frac{2\,\alpha}{2 + \alpha(t_0^2 - t^2)},$$

the denominator of which vanishes at

$$t = \sqrt{\frac{2}{\alpha} + t_0^2}.$$

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- for $|t_0| < T$, ODE has unique solution on

$$D = \{(t, y) \quad | \quad 0 \le t \le T, \quad -Y \le y \le Y\}.$$

- for $\sqrt{\frac{2}{\alpha} + t_0^2} < T$ ODE solution breaks down at $t = \sqrt{\frac{2}{\alpha} + t_0^2}$ on

$$D = \{(t, y) \quad | \quad 0 \le t \le T\}.$$

Well-posed problem

Definition in English: ODE is well-posed if

- A unique ODE solution exists, and
- Small changes (perturbation) to ODE imply small changes to solution.

Well-posed problem

The initial-value problem

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha, \tag{5.2}$$

is said to be a **well-posed problem** if:

- A unique solution, $y(t)$, to the problem exists, and
- There exist constants $\varepsilon_0 > 0$ and $k > 0$ such that for any ε, with $\varepsilon_0 > \varepsilon > 0$, whenever $\delta(t)$ is continuous with $|\delta(t)| < \varepsilon$ for all t in $[a,b]$, and when $|\delta_0| < \varepsilon$, the initial-value problem

$$\frac{dz}{dt} = f(t,z) + \delta(t), \quad a \le t \le b, \quad z(a) = \alpha + \delta_0, \tag{5.3}$$

has a unique solution $z(t)$ that satisfies

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \text{ in } [a,b]. \quad \blacksquare$$

Well-posed problem

Definition in English: ODE is well-posed if

- A unique ODE solution exists, and
- Small changes (perturbation) to ODE imply small changes to solution.

Theorem

Suppose $D = \{(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$. If f is continuous and satisfies a Lipschitz condition in the variable y on the set D, then the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

is well-posed. ∎

Well-posed problem, example

Show that the initial-value problem

$$\frac{dy}{dt} = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5. \tag{5.4}$$

is well posed on $D = \{(t, y) \mid 0 \le t \le 2 \text{ and } -\infty < y < \infty\}$.

Solution: Because

$$\frac{\partial f}{\partial y}(t, y) = 1, \quad \left| \frac{\partial f}{\partial y}(t, y) \right| = 1.$$

$f(t, y) = y - t^2 + 1$ satisfies a Lipschitz condition in y on D with Lipschitz constant 1.Therefore this ODE is well-posed. In fact,

$$y(t) = 1 + t^2 + 2t - \frac{1}{2}e^t.$$

Euler's Method: Initial value ODE to solve

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

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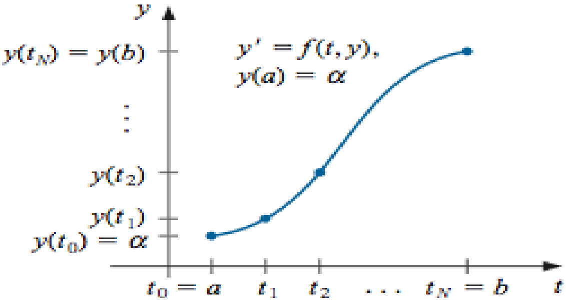
- Choose positive integer N, and select mesh points

$$t_j = a + j\,h, \quad \text{for} \quad j = 0, 1, 2, \cdots N, \quad \text{where } h = (b-a)/N.$$



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- For each j, do 2-term Taylor expansion

  $$y(t_{j+1}) = y(t_j) + h\,y'(t_j) + \frac{h^2}{2} y''(\xi_j), \quad j = 0, 1, \cdots, N - 1.$$

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- Because $y(t)$ satisfies ODE,

  $$y(t_{j+1}) = y(t_j) + h\,f(t_j, y(t_j)) + \frac{h^2}{2} y''(\xi_j), \quad j = 0, 1, \cdots, N-1.$$

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- Ignore error term, set $w_0 = \alpha$,

  $$w_{j+1} = w_j + h\,f(t_j, w_j), \quad j = 0, 1, \cdots, N-1.$$

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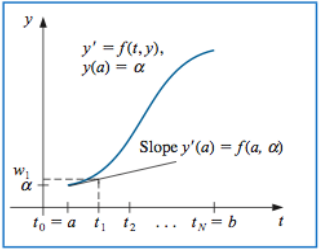
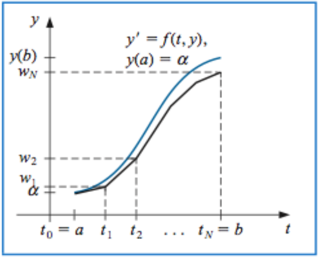


Figure 5.4



Euler's Method: example

Initial Value ODE $\quad \dfrac{dy}{dt} = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$

exact solution $\quad y(t) = (1 + t)^2 - 0.5\, e^t.$

- Choose positive integer $N = 10$, so

$$h = 0.2, \quad t_j = 0.2\, j, \quad \text{for} \quad j = 0, 1, 2, \cdots 10.$$

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- Set $w_0 = 0.5$. For $j = 0, 1, \cdots, 9$,

$$\begin{aligned} w_{j+1} &= w_j + h\,(w_j - t_j^2 + 1) = w_j + 0.2(w_i - 0.04j^2 + 1) \\ &= 1.2w_j - 0.008j^2 + 0.2. \end{aligned}$$

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t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

Error Bounds for Euler's Method

$$D = \{(t, y) \quad | \quad 0 \le t \le T, \quad -Y \le y \le Y\}.$$

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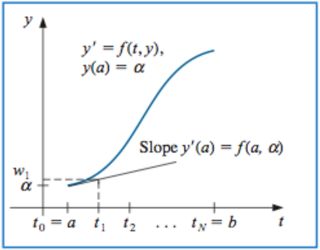

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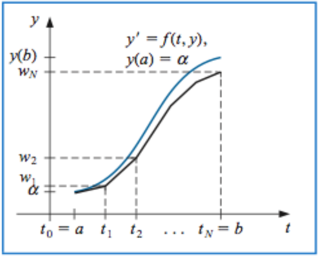

Theorem: Suppose that in the initial value ODE,

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

- $f(t, y)$ is continuous,
- $f(t, y)$ satisfies Lipschitz condition

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &\le L\,|y_1 - y_2| \quad \text{on domain} \\ D &= \{(t, y) \quad | \quad a \le t \le b, \quad -\infty < y < \infty\}. \end{aligned}$$

Let $w_0, w_1, \cdots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then for each $j = 0, 1, \cdots, N$,

$$|y(t_j) - w_j| \le \frac{hM}{2L}\left(e^{L(t_j - a)} - 1\right),$$

where $h = (b - a)/N$, $\quad t_j = a + j\,h$, $M = \mathbf{max}_{t \in [a,b]}\,|y''(t)|$.

Theorem is a mixed bag

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How good is Theorem?

- **Good News**: Euler's method converges:

$$\lim_{N\to\infty} \frac{hM}{2L}\left(e^{L(t_j - a)} - 1\right) = 0.$$

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$$|y(t_j) - w_j| \le \frac{hM}{2L}\left(e^{L(t_j-a)} - 1\right), \qquad (1)$$

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- **Bad News**: N may have to be impossibly large:

$$e^{L(t_N-a)} = e^{L(b-a)} > 10^{43} \quad \text{if} \quad L > 10 \quad \text{and} \quad b-a > 10.$$

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Proof of **Theorem I**

- Mesh points

$$t_j = a + j\,h, \quad \text{for} \quad j = 0, 1, 2, \cdots N, \quad \text{where } h = (b-a)/N.$$

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- For each j, we have 2-term Taylor expansion

$$\begin{aligned} y(t_{j+1}) &= y(t_j) + h\,y'(t_j) + \frac{h^2}{2} y''(\xi_j) \\ &= y(t_j) + h\,f(t_j, y(t_j)) + \frac{h^2}{2} y''(\xi_j), \quad j = 0, 1, \cdots, N-1. \end{aligned}$$

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$$w_{j+1} = w_j + h\,f(t_j, w_j).$$

- Subtraction of two equations:

$$y(t_{j+1}) - w_{j+1} = y(t_j) - w_j + h\left(f(t_j, y(t_j)) - f(t_j, w_j)\right) + \frac{h^2}{2} y''(\xi_j).$$

Proof of **Theorem II**

- Difference of two equations:

$$y(t_{j+1})-w_{j+1} = y(t_j)-w_j+h\left(f(t_j, y(t_j)) - f(t_j, w_j)\right)+\frac{h^2}{2}y''(\xi_j).$$

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- This implies a linear recursion:

$$\begin{aligned}
|y(t_{j+1}) - w_{j+1}| &\leq |y(t_j) - w_j| + h\,|f(t_j, y(t_j)) - f(t_j, w_j)| + \frac{h^2}{2}\left|y''(\xi_j)\right| \\
&\leq |y(t_j) - w_j| + hL\,|y(t_j) - w_j| + \frac{h^2}{2}\,M \\
&= (1 + hL)\,|y(t_j) - w_j| + \frac{h^2}{2}\,M.
\end{aligned}$$

Proof of Theorem II

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$$\begin{aligned}
|y(t_{j+1}) - w_{j+1}| &\leq |y(t_j) - w_j| + h\left|f(t_j, y(t_j)) - f(t_j, w_j)\right| + \frac{h^2}{2}\left|y''(\xi_j)\right| \\
&\leq |y(t_j) - w_j| + hL\,|y(t_j) - w_j| + \frac{h^2}{2}\,M \\
&= (1 + hL)\,|y(t_j) - w_j| + \frac{h^2}{2}\,M.
\end{aligned}$$

- Further simplification for $j = 0, \cdots, N-1$,

$$\begin{aligned}
|y(t_{j+1}) - w_{j+1}| + \frac{hM}{2L} &\leq (1 + hL)\,|y(t_j) - w_j| + \frac{h^2}{2}\,M + \frac{hM}{2L} \\
&= (1 + hL)\left(|y(t_j) - w_j| + \frac{hM}{2L}\right).
\end{aligned}$$

Proof of **Theorem** III

- Solving linear recursion:

$$
\begin{aligned}
|y(t_{j+1}) - w_{j+1}| + \frac{hM}{2L} &\leq (1+hL)\left(|y(t_j) - w_j| + \frac{hM}{2L}\right) \\
&\leq (1+hL)^2\left(|y(t_{j-1}) - w_{j-1}| + \frac{hM}{2L}\right) \\
&\ \ \vdots \\
&\leq (1+hL)^{j+1}\left(|y(t_0) - w_0| + \frac{hM}{2L}\right).
\end{aligned}
$$

- Since $y(t_0) = w_0 = \alpha$,

$$
|y(t_{j+1}) - w_{j+1}| + \frac{hM}{2L} \leq (1+hL)^{j+1}\,\frac{hM}{2L},
$$

it follows that

$$
\begin{aligned}
|y(t_{j+1}) - w_{j+1}| &\leq \frac{hM}{2L}\left((1+hL)^{j+1} - 1\right) \\
&\leq \frac{hM}{2L}\left(e^{L(t_{j+1}-a)} - 1\right).
\end{aligned}
$$

Theorem: Suppose that in the initial value ODE,

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

- $f(t, y)$ is continuous,
- $f(t, y)$ satisfies Lipschitz condition

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &\leq L\,|y_1 - y_2| \quad \text{on domain} \\ D &= \{(t, y) \quad | \quad a \leq t \leq b, \quad -\infty < y < \infty\}. \end{aligned}$$

Let $w_0, w_1, \cdots, w_N$ be the approximations generated by Euler's method for some positive integer N. Then for each $j = 0, 1, \cdots, N$,

$$|y(t_j) - w_j| \leq \frac{hM}{2L}\left(e^{L(t_j - a)} - 1\right),$$

where $h = (b - a)/N, \quad t_j = a + j\,h,\ M = \mathbf{max}_{t \in [a,b]}\,|y''(t)|$.

Euler's Method: example (I)

Initial Value ODE $\quad \frac{dy}{dt} = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$

exact solution $\quad y(t) = (1+t)^2 - 0.5\,e^t.$

- Choose positive integer $N = 10$, so

$$h = 0.2, \quad t_j = 0.2\,j, \quad \text{for} \quad j = 0, 1, 2, \cdots 10.$$

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- Set $w_0 = 0.5$. For $j = 0, 1, \cdots, 9$,

$$\begin{aligned} w_{j+1} &= w_j + h\,(w_j - t_j^2 + 1) = w_j + 0.2(w_i - 0.04j^2 + 1) \\ &= 1.2w_j - 0.008j^2 + 0.2. \end{aligned}$$

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$$\text{Since} \quad y''(t) = 2 - 0.5\, e^t, \quad \frac{\partial f}{\partial y}(t, y) = 1,$$

$$\text{it follows that} \quad \left|y''(t)\right| \le 0.5\, e^2 - 2 \stackrel{def}{=} M, \quad L = 1.$$

Euler's Method: example (II)

Initial Value ODE $\qquad \dfrac{dy}{dt} = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$

exact solution $\qquad y(t) = (1+t)^2 - 0.5\, e^t.$

- Therefore $M = 0.5\, e^2 - 2, L = 1,$

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left(e^{L(t_i - a)} - 1\right) = 0.1\,\left(0.5\, e^2 - 2\right)\,\left(e^{t_i} - 1\right).$$

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t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

$$|y(t_i) - w_i| \leq 0.1\ \left(0.5\, e^2 - 2\right)\ \left(e^{t_i} - 1\right)$$

t_i	Actual Error	Error Bound
0.200000	0.029300	0.037520
0.400000	0.062090	0.083340
0.600000	0.098540	0.139310
0.800000	0.138750	0.207670
1.000000	0.182680	0.291170
1.200000	0.230130	0.393150
1.400000	0.280630	0.517710
1.600000	0.333360	0.669850
1.800000	0.387020	0.855680
2.000000	0.439690	1.082640

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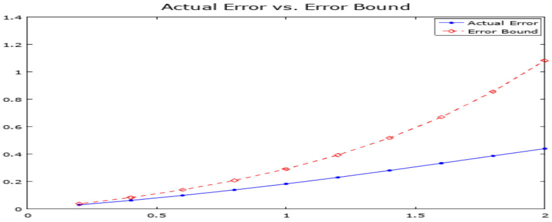

Euler's Method in Finite Precision, I

- Mesh points

$$t_j = a + j\,h, \quad \text{for} \quad j = 0, 1, 2, \cdots N, \quad \text{where } h = (b-a)/N.$$

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$$t_j = a + j\,h, \quad \text{for} \quad j = 0, 1, 2, \cdots N, \quad \text{where } h = (b-a)/N.$$

- For each j, we have 2-term Taylor expansion

$$\begin{aligned} y(t_{j+1}) &= y(t_j) + h\,y'(t_j) + \frac{h^2}{2} y''(\xi_j) \\ &= y(t_j) + h\,f(t_j, y(t_j)) + \frac{h^2}{2} y''(\xi_j), \quad j = 0, 1, \cdots, N-1. \end{aligned}$$

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- Euler's method for each j, in finite precision

$$u_{j+1} = u_j + h\,f(t_j, u_j) + \delta_{j+1},$$

where $|\delta_{j+1}| \le \delta$.

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where $|\delta_{j+1}| \le \delta$.

- Subtraction of two equations:

$$y(t_{j+1}) - u_{j+1} = y(t_j) - u_j + h\left(f(t_j, y(t_j)) - f(t_j, u_j)\right) + \frac{h^2}{2} y''(\xi_j) - \delta_{j+1}.$$

Euler's Method in Finite Precision, II

- Difference of two equations:

$$y(t_{j+1}) - u_{j+1} = y(t_j) - u_j + h\left(f(t_j, y(t_j)) - f(t_j, u_j)\right) + \frac{h^2}{2} y''(\xi_j) - \delta_{j+1}.$$

Euler's Method in Finite Precision, II

- Difference of two equations:

$$y(t_{j+1})-u_{j+1} = y(t_j)-u_j+h\left(f(t_j, y(t_j)) - f(t_j, u_j)\right)+\frac{h^2}{2}y''(\xi_j)-\delta_{j+1}.$$

- This implies a linear recursion:

$$\begin{aligned}
|y(t_{j+1}) - u_{j+1}| &\le |y(t_j)-u_j|+h\left|f(t_j, y(t_j))-f(t_j, u_j)\right|+\frac{h^2}{2}\left|y''(\xi_j)\right|+|\delta_{j+1}| \\
&\le |y(t_j) - u_j| + hL\,|y(t_j) - u_j| + \frac{h^2}{2}\,M + \delta \\
&= (1 + hL)\,|y(t_j) - u_j| + \frac{h^2}{2}\,M + \delta.
\end{aligned}$$

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- Difference of two equations:

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&= (1 + hL)\,|y(t_j) - u_j| + \frac{h^2}{2}\,M + \delta.
\end{aligned}$$

- Further simplification for $j = 0, \cdots, N-1$,

$$\begin{aligned}
|y(t_{j+1}) - u_{j+1}| + \frac{hM}{2L} + \frac{\delta}{hL} &\leq (1 + hL)\,|y(t_j) - u_j| + \frac{h^2}{2}\,M + \frac{hM}{2L} + \delta + \frac{\delta}{hL} \\
&= (1 + hL)\left(|y(t_j) - u_j| + \frac{hM}{2L} + \frac{\delta}{hL}\right).
\end{aligned}$$

Euler's Method in Finite Precision, III

- Solving linear recursion:

$$
\begin{aligned}
|y(t_{j+1}) - u_{j+1}| + \frac{hM}{2L} + \frac{\delta}{hL} &\leq (1 + hL)\left(|y(t_j) - u_j| + \frac{hM}{2L} + \frac{\delta}{hL}\right) \\
&\vdots \\
&\leq (1 + hL)^{j+1}\left(|y(t_0) - u_0| + \frac{hM}{2L} + \frac{\delta}{hL}\right).
\end{aligned}
$$

- Assume $|y(t_0) - u_0| = |\alpha - u_0| \leq \delta$,

$$
|y(t_{j+1}) - u_{j+1}| + \frac{hM}{2L} + \frac{\delta}{hL} \leq (1 + hL)^{j+1}\left(\delta + \frac{hM}{2L} + \frac{\delta}{hL}\right),
$$

it follows that

$$
\begin{aligned}
|y(t_{j+1}) - u_{j+1}| &\leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left((1 + hL)^{j+1} - 1\right) + \delta\,(1 + hL)^{j+1} \\
&\leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_{j+1} - a)} - 1\right) + \delta\, e^{L(t_{j+1} - a)}.
\end{aligned}
$$

Euler's Method in Finite Precision, IV

- Error bound in finite precision

$$|y(t_{j+1}) - u_{j+1}| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_{j+1}-a)} - 1\right) + \delta\, e^{L(t_{j+1}-a)}.$$

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- Value of h can't be too small:

$$\lim_{h\to 0}\left(\frac{hM}{2} + \frac{\delta}{h}\right) = \infty.$$

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- "Optimal" value of h

$$h_{\mathbf{opt}} = \sqrt{\frac{2\delta}{M}}, \quad \text{and} \quad \left(\frac{h_{\mathbf{opt}}\,M}{2} + \frac{\delta}{h_{\mathbf{opt}}}\right) = \sqrt{2\delta\, M}.$$

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- "Practical" value of $h \gg h_{\mathbf{opt}}$.

Local Truncation Error for a general difference method

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

Definition: The difference method

$$\begin{aligned} w_0 &= \alpha, \\ w_{j+1} &= w_j + h\phi\left(t_j, w_j\right), \quad \text{for} \quad j = 0, 1, \cdots, N-1 \end{aligned}$$

has **local truncation error**

$$\begin{aligned} \tau_{j+1}(h) &\stackrel{def}{=} \frac{y(t_{j+1}) - \left(y(t_j) + h\,\phi\left(t_j, y(t_j)\right)\right)}{h} \\ &= \frac{y(t_{j+1}) - y(t_j)}{h} - \phi\left(t_j, y(t_j)\right). \end{aligned}$$

Local Truncation Error for Euler's method

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Euler's method

$$\begin{aligned} w_0 &= \alpha, \\ w_{j+1} &= w_j + hf\left(t_j, w_j\right), \quad \text{for} \quad j = 0, 1, \cdots, N-1 \end{aligned}$$

has **local truncation error**

$$\begin{aligned} \tau_{j+1}(h) &= \frac{y(t_{j+1}) - y(t_j)}{h} - f\left(t_j, y(t_j)\right) \\ &= \frac{y(t_{j+1}) - y(t_j)}{h} - y'(t_j) = \frac{h}{2} y''(\xi_j), \quad \xi_j \in (t_j, tj+1). \end{aligned}$$

This implies

$$|\tau_{j+1}(h)| \le \frac{h}{2} \mathbf{max}_{t \in [a,b]} \left|y''(t)\right| \stackrel{def}{=} \frac{h\,M}{2}, \quad \text{a first order method.}$$

n-th Order Taylor Methods:

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

n-th Order Taylor Methods:

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

- $(n+1)$-term Taylor expansion

$$\begin{aligned} y(t_{j+1}) &= \quad y(t_j) + h y'(t_j) + \frac{h^2}{2} y''(t_j) + \cdots + \frac{h^n}{n!} y^{(n)}(t_j) \\ &\quad + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_j). \end{aligned}$$

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- On the other hand,

$$\begin{aligned} y'(t_j) &= f(t_j, y(t_j)), \quad y''(t_j) = f'(t_j, y(t_j)), \cdots \\ y^{(n)}(t_j) &= f^{(n-1)}(t_j, y(t_j)). \end{aligned}$$

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- n-th order $\boxed{\text{Taylor method}}$

$$\begin{aligned} w_0 &= \alpha, \\ w_{j+1} &= w_j + h\mathbf{T}^{(n)}(t_j, w_j), \quad j = 0, 1, \cdots, N-1, \end{aligned}$$

where $\quad \mathbf{T}^{(n)}(t_j, w_j) = f(t_j, w_j) + \frac{h}{2} f'(t_j, w_j) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_j, w_j).$

Example: Second Order Taylor Method:

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

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$$\begin{aligned} f'(t, y(t)) &= \frac{d}{dt} f(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) y'(t) \\ &= \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))\, f(t, y(t)). \end{aligned}$$

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$$\begin{aligned} f'(t, y(t)) &= \frac{d}{dt} f(t, y(t)) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) y'(t) \\ &= \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))\, f(t, y(t)). \end{aligned}$$

- Second order Taylor method in explicit form

$$w_{j+1} = w_j + h\left(f(t_j, w_j) + \frac{h}{2}\left(\frac{\partial f}{\partial t}(t_j, w_j) + \frac{\partial f}{\partial y}(t_j, w_j)\, f((t_j, w_j))\right)\right).$$

Example: Second Order Taylor Method:

$$\frac{dy}{dt} = f(t, y), \quad f(t, y) = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

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$$\frac{\partial f}{\partial t}(t,y(t)) = -2t, \quad \frac{\partial f}{\partial y}(t,y(t)) = 1.$$

Thus,

$$\begin{aligned} w_{j+1} &= w_j + h\left(w_j - t_j^2 + 1 + \frac{h}{2}\left(-2t_j + w_j - t_j^2 + 1\right)\right) \\ &= \left(1 + h + \frac{h^2}{2}\right) w_j + \left(h + \frac{h^2}{2}\right)\left(1 - t_j^2\right) - h^2 t_j. \end{aligned}$$

Euler's Method vs. Second Order Taylor Method: $N = 10$

$$\frac{dy}{dt} = f(t, y), \quad f(t, y) = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

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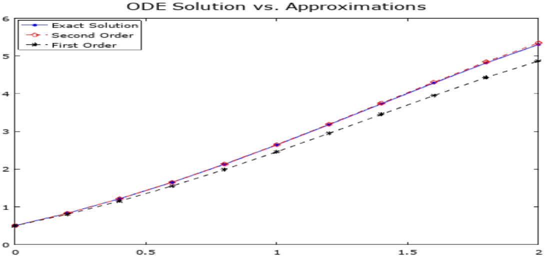
t_i	Euler's Method	Taylor Method	Exact Solution
0.00000	0.50000	0.50000	0.50000
0.20000	0.80000	0.83000	0.82930
0.40000	1.15200	1.21580	1.21409
0.60000	1.55040	1.65208	1.64894
0.80000	1.98848	2.13233	2.12723
1.00000	2.45818	2.64865	2.64086
1.20000	2.94981	3.19135	3.17994
1.40000	3.45177	3.74864	3.73240
1.60000	3.95013	4.30615	4.28348
1.80000	4.42815	4.84630	4.81518
2.00000	4.86578	5.34768	5.30547

Euler's Method vs. Second Order Taylor Method: $N = 10$

- Second order method looks more accurate.



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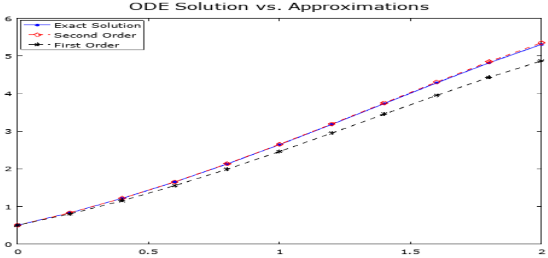
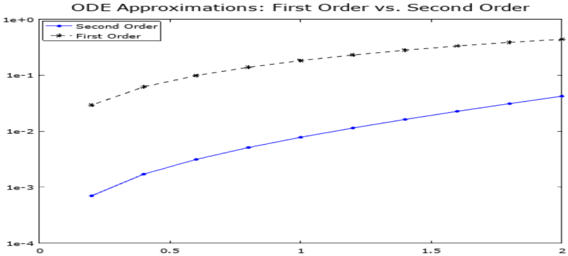


- Second order method has much smaller errors.



Runge-Kutta methods

With orders of Taylor methods yet without derivatives of $f(t, y(t))$

First order Taylor expansion in two variables

Theorem: Suppose that $f(t,y)$ and all its partial derivatives are

continuous on $D \stackrel{def}{=} \{(t,y) \mid \quad a \leq t \leq b, \quad c \leq y \leq d.\}$

Let $(t_0, y_0), (t,y) \in D, \Delta_t = t - t_0, \Delta_y = y - y_0$. **Then**

$$f(t,y) = P_1(t,y) + R_1(t,y), \quad \text{where}$$

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$$\begin{aligned}
P_1(t,y) &= f(t_0,y_0) + \Delta_t \frac{\partial f}{\partial t}(t_0,y_0) + \Delta_y \frac{\partial f}{\partial y}(t_0,y_0), \\
R_1(t,y) &= \frac{\Delta_t^2}{2} \frac{\partial^2 f}{\partial t^2}(\xi,\mu) + \Delta_t \, \Delta_y \frac{\partial^2 f}{\partial t \partial y}(\xi,\mu) + \frac{\Delta_y^2}{2} \frac{\partial^2 f}{\partial y^2}(\xi,\mu),
\end{aligned}$$

for some point $(\xi, \mu) \in D$.

Second order Taylor expansion in two variables

Theorem: Suppose that $f(t, y)$ and all its partial derivatives are

$$\text{continuous on} \quad D \stackrel{def}{=} \{(t, y) \mid \quad a \leq t \leq b, \quad c \leq y \leq d.\}$$

Let $(t_0, y_0), (t, y) \in D, \Delta_t = t - t_0, \Delta_y = y - y_0$. **Then**

$$f(t, y) = P_2(t, y) + R_2(t, y), \quad \text{where}$$

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$$\begin{aligned} P_2(t, y) \;\; = \;\; & f(t_0, y_0) + \left(\Delta_t \frac{\partial f}{\partial t}(t_0, y_0) + \Delta_y \frac{\partial f}{\partial y}(t_0, y_0) \right) \\ & + \left(\frac{\Delta_t^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + \Delta_t \, \Delta_y \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) + \frac{\Delta_y^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right), \end{aligned}$$

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$$f(t,y) = P_2(t,y) + R_2(t,y), \quad \text{where}$$

$$\begin{aligned} P_2(t,y) &= f(t_0,y_0) + \left(\Delta_t \frac{\partial f}{\partial t}(t_0,y_0) + \Delta_y \frac{\partial f}{\partial y}(t_0,y_0)\right) \\ &\quad + \left(\frac{\Delta_t^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + \Delta_t\,\Delta_y \frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) + \frac{\Delta_y^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right), \\ R_2(t,y) &= \frac{1}{3!}\left(\sum_{j=0}^{3}\binom{3}{j}\Delta_t^{3-j}\,\Delta_y^{j}\frac{\partial^3 f}{\partial t^{3-j}\partial y^j}(\xi,\mu)\right) \end{aligned}$$

Example: Second order Taylor expansion in two variables

$$\begin{aligned} f(t,y) &= \mathbf{exp}\left(-\frac{(t-2)^2}{4}-\frac{(y-3)^2}{4}\right)\mathbf{cos}(2t+y-7) \\ &= P_2(t,y)+R_2(t,y), \quad \text{with} \quad (t_0,y_0)=(2,3). \end{aligned}$$

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$$\begin{aligned} P_2(t,y) &= f(t_0,y_0) + \left(\Delta_t \frac{\partial f}{\partial t}(t_0,y_0) + \Delta_y \frac{\partial f}{\partial y}(t_0,y_0)\right) \\ &+ \left(\frac{\Delta_t^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + \Delta_t\,\Delta_y \frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) + \frac{\Delta_y^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right) \end{aligned}$$

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&\quad + \left(\frac{\Delta_t^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + \Delta_t\,\Delta_y \frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) + \frac{\Delta_y^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right) \\
&= 1 - \frac{9}{4}\Delta_t^2 - 2\Delta_t\,\Delta_y - \frac{3}{4}\Delta_y^2.
\end{aligned}
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Let $\Delta_t = t-2, \Delta_y = y-3,$

$$\begin{aligned} P_2(t,y) &= f(t_0,y_0)+\left(\Delta_t \frac{\partial f}{\partial t}(t_0,y_0)+\Delta_y \frac{\partial f}{\partial y}(t_0,y_0)\right) \\ &\quad +\left(\frac{\Delta_t^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0)+\Delta_t\,\Delta_y \frac{\partial^2 f}{\partial t \partial y}(t_0,y_0)+\frac{\Delta_y^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right) \\ &= 1-\frac{9}{4}\Delta_t^2-2\Delta_t\,\Delta_y-\frac{3}{4}\Delta_y^2. \end{aligned}$$

so $\quad f(t,y) \approx P_2(t,y) = 1-\frac{9}{4}(t-2)^2-2(t-1)(y-3)-\frac{3}{4}(y-3)^2.$

$$
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\end{aligned}
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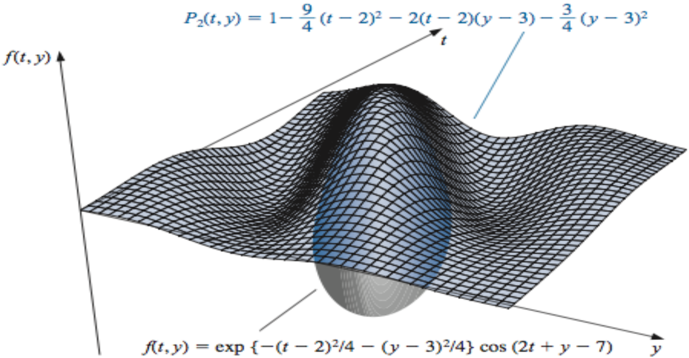
$$



Approximation good near $(2,3)$, bad elsewhere

Taylor's Theorem in two variables

Theorem: Suppose that $f(t,y)$ and all its partial derivatives are

$$\text{continuous on} \quad D \stackrel{def}{=} \{(t,y) \,|\, \quad a \leq t \leq b, \quad c \leq y \leq d.\}$$

Let $(t_0, y_0), (t, y) \in D, \Delta_t = t - t_0, \Delta_y = y - y_0$. **Then** for $n \geq 1$,

$$f(t,y) = P_n(t,y) + R_n(t,y), \quad \text{where}$$

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P_n(t,y) &= f(t_0,y_0) + \left(\Delta_t \frac{\partial f}{\partial t}(t_0,y_0) + \Delta_y \frac{\partial f}{\partial y}(t_0,y_0)\right) \\
&\quad + \left(\frac{\Delta_t^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + \Delta_t\,\Delta_y \frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) + \frac{\Delta_y^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\right) \\
&\quad + \cdots + \frac{1}{n!}\left(\sum_{j=0}^{n} \binom{n}{j} \Delta_t^{n-j}\,\Delta_y^{j} \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right), \\
R_n(t,y) &= \frac{1}{(n+1)!}\left(\sum_{j=0}^{n+1} \binom{n+1}{j} \Delta_t^{n+1-j}\,\Delta_y^{j} \frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi,\mu)\right)
\end{aligned}$$

Runge-Kutta Method of Order Two (I)

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

Runge-Kutta Method of Order Two (I)

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha.$$

- Second order Taylor method

$$w_0 = \alpha,$$
$$w_{j+1} = w_j + h\mathbf{T}^{(2)}(t_j, w_j), \quad j = 0, 1, \cdots, N-1, \quad \text{where}$$

$$\begin{aligned}
\mathbf{T}^{(2)}(t,y) &= f(t,y) + \frac{h}{2} f'(t,y) \\
&= f(t,y) + \frac{h}{2}\left(\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\,\frac{dy}{dt}\right) \\
&= f(t,y) + \frac{h}{2}\left(\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\,f(t,y)\right) \\
&= f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right)
\end{aligned}$$

Runge-Kutta Method of Order Two (II)

- Second order Taylor method

$$w_0 = \alpha,$$
$$w_{j+1} = w_j + h\mathbf{T}^{(2)}(t_j, w_j), \quad j = 0, 1, \cdots, N-1, \quad \text{where}$$

$$\mathbf{T}^{(2)}(t, y) = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) - R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right),$$

- From first order Taylor expansion,

$$R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right) = \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}(\xi, \mu) + \frac{h^2 f(t, y)}{4}\Delta_t \Delta_y \frac{\partial^2 f}{\partial t \partial y}(\xi, \mu) + \frac{h^2 f^2(t, y)}{8}\frac{\partial^2 f}{\partial y^2}(\xi, \mu) = O(h^2).$$

Method remains second order after dropping $R_1\left(t + \frac{h}{2}, y + \frac{h}{2}f(t, y)\right)$

Runge-Kutta Method of Order Two (III)

- **Midpoint Method**

$$
\begin{aligned}
w_0 &= \alpha, \\
w_{j+1} &= w_j + h\,f\left(t_j + \frac{h}{2}, w_j + \frac{h}{2} f(t_j, w_j)\right), \quad j = 0, 1, \cdots, N-1.
\end{aligned}
$$

- Two function evaluations for each j,
- Second order accuracy.

No need for derivative calculations

General 2nd order Runge-Kutta Methods

- $w_0 = \alpha$; for $j = 0, 1, \cdots, N-1$,

$$w_{j+1} = w_j + h\left(a_1 f(t_j, w_j) + a_2 f\left(t_j + \alpha_2, w_j + \delta_2 f(t_j, w_j)\right)\right).$$

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- $w_0 = \alpha$; for $j = 0, 1, \cdots, N-1$,

$$w_{j+1} = w_j + h\left(a_1 f(t_j, w_j) + a_2 f\left(t_j + \alpha_2, w_j + \delta_2 f(t_j, w_j)\right)\right).$$

- Two function evaluations for each j,
- Want to choose $a_1, a_2, \alpha_2, \delta_2$ for highest possible order of accuracy.

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- $w_0 = \alpha$; for $j = 0, 1, \cdots, N-1$,

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- Two function evaluations for each j,
- Want to choose $a_1, a_2, \alpha_2, \delta_2$ for highest possible order of accuracy.

local truncation error

$$\begin{aligned}
\tau_{j+1}(h) &= \frac{y(t_{j+1}) - y(t_j)}{h} - \left(a_1 f(t_j, y(t_j)) + a_2 f\left(t_j + \alpha_2, y(t_j) + \delta_2 f(t_j, y(t_j))\right)\right) \\
&= y'(t_j) + \frac{h}{2} y''(t_j) + O(h^2) \\
&\quad - \left((a_1 + a_2) f(t_j, y(t_j)) + a_2\,\alpha_2 \frac{\partial f}{\partial t}(t_j, y(t_j))\right. \\
&\quad + \left. a_2\,\delta_2 f(t_j, y(t_j)) \frac{\partial f}{\partial y}(t_j, y(t_j)) + O(h^2)\right).
\end{aligned}$$

local truncation error

$$\begin{aligned}\tau_{j+1}(h) = \; & y'(t_j) + \frac{h}{2} y''(t_j) + O(h^2) \\ & - \Bigg((a_1 + a_2)\, f(t_j, y(t_j)) + a_2\, \alpha_2\, \frac{\partial f}{\partial t}(t_j, y(t_j)) \\ & \qquad + \quad a_2\, \delta_2 f(t_j, y(t_j)) \frac{\partial f}{\partial y}(t_j, y(t_j)) + O(h^2) \Bigg) \,.\end{aligned}$$

where $y'(t_j) = f(t_j, y(t_j))$,

local truncation error

$$\tau_{j+1}(h) = \; y'(t_j) + \frac{h}{2} y''(t_j) + O(h^2)$$
$$- \Big((a_1 + a_2)\, f(t_j, y(t_j)) + a_2\, \alpha_2\, \frac{\partial f}{\partial t}(t_j, y(t_j))$$
$$+ \quad a_2\, \delta_2 f(t_j, y(t_j)) \frac{\partial f}{\partial y}(t_j, y(t_j)) + O(h^2) \Big)\,.$$

where $y'(t_j) = f(t_j, y(t_j))$,

$$y''(t_j) = \frac{d\, f}{d\, t}(t_j, y(t_j)) = \frac{\partial f}{\partial t}(t_j, y(t_j)) + f(t_j, y(t_j)) \frac{\partial f}{\partial y}(t_j, y(t_j)).$$

For any choice with

$$a_1 + a_2 = 1, \quad a_2\, \alpha_2 = a_2\, \delta_2 = \frac{h}{2},$$

we have a second order method

$$\tau_{j+1}(h) = O(h^2).$$

local truncation error

$$\tau_{j+1}(h) = y'(t_j) + \frac{h}{2} y''(t_j) + O(h^2)$$
$$- \Big((a_1 + a_2)\, f(t_j, y(t_j)) + a_2\, \alpha_2\, \frac{\partial f}{\partial t}(t_j, y(t_j))$$
$$+ \quad a_2\, \delta_2 f(t_j, y(t_j)) \frac{\partial f}{\partial y}(t_j, y(t_j)) + O(h^2) \Big) .$$

where $y'(t_j) = f(t_j, y(t_j))$,

$$y''(t_j) = \frac{d\, f}{d\, t}(t_j, y(t_j)) = \frac{\partial f}{\partial t}(t_j, y(t_j)) + f(t_j, y(t_j)) \frac{\partial f}{\partial y}(t_j, y(t_j)).$$

For any choice with

$$a_1 + a_2 = 1, \quad a_2\, \alpha_2 = a_2\, \delta_2 = \frac{h}{2},$$

we have a second order method

$$\tau_{j+1}(h) = O(h^2).$$

Four parameters, three equations

General 2nd order Runge-Kutta Methods

$w_0 = \alpha$; for $j = 0, 1, \cdots, N-1$,

$$w_{j+1} = w_j + h\left(a_1 f(t_j, w_j) + a_2 f\left(t_j + \alpha_2, w_j + \delta_2 f(t_j, w_j)\right)\right).$$

$$a_1 + a_2 = 1, \quad a_2\,\alpha_2 = a_2\,\delta_2 = \frac{h}{2},$$

- **Midpoint method**: $a_1 = 0, a_2 = 1, \alpha_2 = \delta_2 = \frac{h}{2}$,

$$w_{j+1} = w_j + h\,f\left(t_j + \frac{h}{2}, w_j + \frac{h}{2} f(t_j, w_j)\right).$$

- **Modified Euler method**: $a_1 = a_2 = \frac{1}{2}, \alpha_2 = \delta_2 = h$,

$$w_{j+1} = w_j + \frac{h}{2}\left(f(t_j, w_j) + f\left(t_{j+1}, w_j + hf(t_j, w_j)\right)\right).$$

3rd order Runge-Kutta Method (rarely used in practice)

$$w_0 = \alpha;$$
for $\quad j = 0, 1, \cdots, N-1,$

$$w_{j+1} = w_j + \frac{h}{4}\left(f(t_j, w_j) + 3f\left(t_j + \frac{2h}{3}, w_j + \frac{2h}{3} f(t_j + \frac{h}{3}, w_j + \frac{h}{3} f(t_j, w_j))\right)\right)$$

$$\stackrel{def}{=} w_j + h\,\phi(t_j, w_j).$$

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$$\stackrel{def}{=} w_j + h\,\phi(t_j, w_j).$$

local truncation error

$$\tau_{j+1}(h) = \frac{y(t_{j+1}) y(t_j)}{h} - \phi(t_j, y(t_j)) = O(h^3).$$

4th order Runge-Kutta Method

$w_0 = \alpha;$
for $\quad j = 0, 1, \cdots, N-1,$

$$\begin{aligned}
k_1 &= h\, f(t_j, w_j), \\
k_2 &= h\, f\left(t_j + \frac{h}{2}, w_j + \frac{1}{2} k_1\right), \\
k_3 &= h\, f\left(t_j + \frac{h}{2}, w_j + \frac{1}{2} k_2\right), \\
k_4 &= h\, f\left(t_{j+1}, w_j + k_3\right), \\
w_{j+1} &= w_j + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).
\end{aligned}$$

4 function evaluations per step

Example

Initial Value ODE $\quad \dfrac{dy}{dt} = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5,$

exact solution $\quad y(t) = (1 + t)^2 - 0.5\, e^t.$

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t_i	Exact	Euler $h = 0.025$	Modified Euler $h = 0.05$	Runge-Kutta Order Four $h = 0.1$
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.6574145	0.6554982	0.6573085	0.6574144
0.2	0.8292986	0.8253385	0.8290778	0.8292983
0.3	1.0150706	1.0089334	1.0147254	1.0150701
0.4	1.2140877	1.2056345	1.2136079	1.2140869
0.5	1.4256394	1.4147264	1.4250141	1.4256384

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