Math128A: Numerical Analysis Sample Final

This is a closed book, closed notes exam, with the exception of a one-page/one-side cheat sheet. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Do as much as you can. Partial solutions will get partial credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so.

Your Name: ______________________

Your SID: ______________________
1. (10 points)

(a) Describe a method to evaluate \( \int_{1}^{\infty} \frac{e^{-x}}{x^2} \, dx \). No actual calculation is required.

(b) Evaluate

\[
\int_{-1}^{1} \int_{-2}^{2} \left( x^2 + y^2 + xy \right) \, dxdy.
\]
2. (10 points)

(a) From the Taylor expansion of a function $f(x)$, derive a first-order approximation to $f'(x)$.

(b) Use Richardson’s extrapolation method to find a 3 point 2nd order formula.
3. (10 points)  
   
   (a) Let $A$ and $B$ be $n \times n$ matrices. Prove or find a counter example: If $AB = 0$ then $A = 0$ or $B = 0$.  
   
   (b) Let $A$ and $B$ be $n \times n$ matrices. Prove or find a counter example: If $AB = 0$ then $\det(A) = 0$ or $\det(B) = 0$.  

4. (10 points) Consider the iteration

\[ x_{k+1} = 2x_k - \alpha x_k^2, \quad k = 0, 1, \ldots, \]

where \( \alpha > 0 \) is given. Show that the iteration converges quadratically to \( 1/\alpha \) for any initial guess \( x_0 \) satisfying \( 0 < x_0 < 2/\alpha \).
5. (10 points)

(a) For a function $f$ and distinct points $\alpha$, $\beta$, and $\gamma$, define what is meant by $f[\alpha, \beta, \gamma]$.

(b) Find the Lagrange form of the polynomial $P(x)$ which interpolates $f(x) = \frac{4x}{x + 1}$ at 0, 1, and 3.
6. (10 points) For the following linear system

\[
\begin{align*}
x - \alpha y &= 1, \\
\alpha x - y &= 1,
\end{align*}
\]

describe values of \( \alpha \) for which the system has an infinite number of solutions, no solutions, and exactly one solution, and find the solution when it is unique.
7. (10 points) Determine the free cubic spline that approximates the data \( f(-1) = 1, \ f(0) = 0 \) and \( f(1) = 1 \).
8. (10 points)
   (a) Define what is meant by the local truncation error, and the local order, for a single-step method for solving the ODE’s.
   (b) Derive a specific Runge-Kutta method of local order 2. Show your work.
9. (10 points)

(a) Let $A$ be an $n \times n$ non-singular lower triangular matrix. Describe the forward substitution algorithm for solving $Ax = b$.

(b) Solve $Ax = b$ where

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
10. (10 points) Find the region of absolute stability of the following methods

(a) The Modified Euler method: \( w_0 = \alpha, \)
\[
w_{j+1} = w_j + \frac{h}{2} \left( f(t_j, w_j) + f(t_{j+1}, w_j + h f(t_j, w_j)) \right), \quad j = 0, 1, \cdots, n.
\]

(b) Adams-Bashforth Two-Step Explicit Method: \( w_0 = \alpha, w_1 = \alpha_1, \)
\[
w_{j+1} = w_j + \frac{h}{2} \left( 3f(t_j, w_j) - f(t_{j-1}, w_{j-1}) \right), \quad j = 1, 2, \cdots, n.
\]