1. Find $f'(x)$ where $f(x) = 3^x$.

   Solution: If we take ln in both side of this equation and differentiate it, we get

$$\begin{aligned} & \ln f(x) = \ln 3^x \\ \therefore\quad & \ln f(x) = x \ln 3 \\ \therefore\quad & \frac{d}{dx}(\ln f(x)) = \frac{d}{dx}(x \ln 3) \\ \therefore\quad & \frac{1}{f(x)} f'(x) = \ln 3 \\ \therefore\quad & f'(x) = f(x) \ln 3 \\ \therefore\quad & f'(x) = 3^x \ln 3 \end{aligned}$$

2. Find $f'(x)$ where $f(x) = (2x)^{1/x}$.

   Solution: Similarly to the previous problem, we have

$$\begin{aligned} & \ln f(x) = \frac{1}{x} \ln 2x \\ \therefore\quad & \frac{d}{dx}(\ln f(x)) = \frac{d}{dx}\left(\frac{1}{x} \ln 2x\right) \\ \therefore\quad & \frac{1}{f(x)} f'(x) = \frac{-1}{x^2} \ln 2x + \frac{1}{x}\frac{2}{2x} \\ \therefore\quad & f'(x) = f(x)\left(\frac{-\ln(2x) + 1}{x^2}\right) \\ \therefore\quad & f'(x) = (2x)^{1/x}\left(\frac{-\ln(2x) + 1}{x^2}\right) \end{aligned}$$

3. Find $f'(x)$ where $f(x) = (x)^{e^x}$.

   Solution: We have

$$\begin{aligned} & \ln f(x) = e^x \ln x \\ \therefore\quad & \frac{d}{dx}(\ln f(x)) = \frac{d}{dx}(e^x \ln x) \\ \therefore\quad & \frac{1}{f(x)} f'(x) = e^x \ln x + e^x \frac{1}{x} \\ \therefore\quad & f'(x) = (x)^{e^x}\left(e^x \ln x + e^x \frac{1}{x}\right) \end{aligned}$$

4. Find $f'(x)$ where $f(x) = (x+1)^{1-x}$.

   Solution: We have

$$\begin{aligned} & \ln f(x) = (1-x)\ln(x+1) \\ \therefore\quad & \frac{d}{dx}(\ln f(x)) = \frac{d}{dx}((1-x)\ln(x+1)) \\ \therefore\quad & \frac{1}{f(x)} f'(x) = -\ln(x+1) + (1-x)\frac{1}{x+1} \\ \therefore\quad & f'(x) = (x+1)^{1-x}\left(-\ln(x+1) + \frac{1-x}{x+1}\right) \end{aligned}$$

5. Evaluate $\int (2x+7)^5 dx$.

   Solution:
$$\int (2x+7)^5 dx = \frac{(2x+7)^6}{6*2} + C = \frac{(2x+7)^6}{12} + C$$
   where $C \in \mathbb{R}$.

6. Evaluate $\int (ax+b)^k dx$ for all $k \in \mathbb{R}$ (assume $a \neq 0$).

   Solution: If $k \neq 1$ we have
$$\int (ax+b)^k dx = \frac{(ax+b)^{k+1}}{(k+1)a} + C$$
   where $C \in \mathbb{R}$.

   For $k = -1$ we have
$$\int (ax+b)^{-1} dx = \ln|ax+b| + C$$
   where $C \in \mathbb{R}$.

7. Find f such that $f'(x) = e^{5x}$.

   Solution: $f(x) = \dfrac{e^{5x}}{5} + C$, where $C \in \mathbb{R}$.

8. Find f such that $f'(x) = \dfrac{2}{x^{3/2}}$ and $f(1) = 5$.

   Solution: Since $f'(x) = \dfrac{2}{x^{3/2}}$, we have $f(x) = 2\dfrac{x^{-3/2+1}}{-3/2+1} + C = -4x^{-1/2} + C$, for some constant $C \in \mathbb{R}$. Imposing $f(1) = 5$ we get
$$5 = -4(1)^{-1/2} + C \quad \Rightarrow \quad 5 = -4 + C \quad \Rightarrow \quad C = 9.$$
   Therefore, $f(x) = -4x^{-1/2} + 9$.

9. Find f such that $f''(x) = \frac{1}{x^2}$, $f(2) = 0$ and $f(1) = 1$.

   Solution: Here we need to integrate twice.

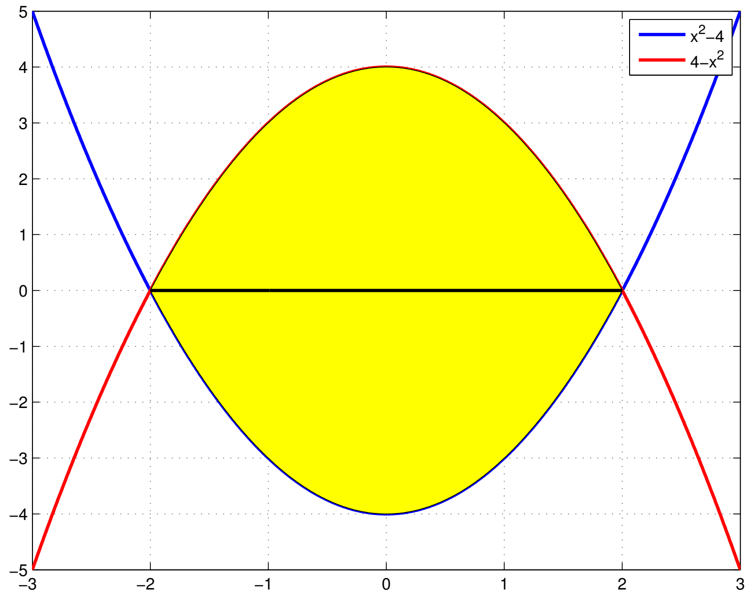
   Since $f''(x) = \frac{1}{x^2}$, then integrating the equation once we get $f'(x) = \frac{x^{-1}}{-1} + C_1 = -\frac{1}{x} + C_1$, for some constant C_1.

   Now since $f'(x) = -\frac{1}{x} + C_1$, integrating this equation we get $f(x) = -\ln|x| + C_1 x + C_2$, for some constants C_1 and C_2.

   Using $f(x) = -\ln|x| + C_1 x + C_2$, and setting $f(2) = 0$ and $f(1) = 1$ we get
$$\begin{cases} -\ln 2 + 2C_1 + C_2 = 0 \\ C_1 + C_2 = 1 \end{cases}$$
$$\therefore \quad \begin{cases} 2C_1 + C_2 = \ln 2 & (I) \\ C_1 + C_2 = 1 & (II) \end{cases}$$
$$\therefore \quad C_1 = \ln 2 - 1 \quad \big(\text{from } (I) - (II)\big)$$
$$\therefore \quad C_2 = 2 - \ln 2 \quad \big(\text{from } (II)\big)$$

Therefore $f(x) = -\ln|x| + (\ln 2 - 1)x + (2 - \ln 2)$ (check that this f satisfy all conditions $f''(x) = \frac{1}{x^2}$, $f(2) = 0$ and $f(1) = 1$).

10. Find the area enclosed by the curves $y = x^2 - 4$ and $y = 4 - x^2$.



By symmetry, we note that the area A enclosed by the curves $y = x^2 - 4$ and $y = 4 - x^2$ is

$$A = 2\int_{-2}^{2} 4 - x^2 dx = 2\left(4x - \frac{x^3}{3}\right)\Big|_{x=-2}^{x=2} = 2\left(8 - \frac{8}{3} - (-8 + \frac{8}{3})\right) = \frac{64}{3}$$

You could also have written $A = \int_{-2}^{2} 4 - x^2 dx - \int_{-2}^{2} x^2 - 4dx$.

11. (in the worksheet given in class the function was $x^2 - 4x + 5$) Find the area under the curve $y(x) = x^2 - 6x + 5$ from $x = -1$ to $x = 3$.

We note that the area is given by

$$\begin{aligned} A &= \int_{-1}^{1}(x^2-6x+5)dx - \int_{1}^{3}(x^2-6x+5)dx = \\ &= (x^3/3-3x^2+5x)\Big|_{x=-1}^{x=1} - (x^3/3-3x^2+5x)\Big|_{x=1}^{x=3} = 16 \end{aligned}$$

12. (Textbook, p. 289, ex. 13) The population of a certain country is growing exponentially. The total population (in millions) in t years is given by the function $P(t)$. Match each of the following answers with its corresponding question.

Answers:

(a) Solve $P(t) = 2$ for t.
(b) $P(2)$
(c) $P'(2)$
(d) Solve $P'(t) = 2$ for t
(e) $y' = ky$
(f) Solve $P(t) = 2P(0)$ for t
(g) P_0e^{kt}, $k > 0$
(h) P(0)

Questions:

(I) How fast will be the population growing in 2 years?
(II) Give the general formula of the function $P(t)$.
(III) How long will it take for the current population to double?
(IV) What will be the size of the population in 2 years?

(V) What is the initial size of the population?

(VI) When will be the size of the population 2 million?

(VII) When will be the population growing at the rate of 2 million people per year?

(VIII) Give the differential equation satisfied by $P(t)$.

Solution: (a)-(VI); (b)-(IV); (c)-(I); (d)-(VII); (e)-(VIII); (f)-(III); (g)-(II); (h)-(V)