Midterm 2 Problem List
Math 16a, Spring 2009

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—solutions—

1. **Find the maximum value of $a\sqrt{b}$ given that $a > 0$, $b > 0$ and $a + b = 1$.**

   Since $a + b = 1$, we have $a = 1 - b$. Hence we seek the maximum value of

   $$f(b) = (1 - b)\sqrt{b} = b^{1/2} - b^{3/2}$$

   for $b > 0$. Differentiating using the power rule:

   $$f'(b) = (1/2)b^{(1/2)-1} - (3/2)b^{(3/2)-1} = (1/2)b^{-1/2} - (3/2)b^{1/2}.$$

   The maximum value of $f(b)$ on the interval $(0, \infty)$ must occur at a point where $f'(b) = 0$. Set

   $$f'(b) = 0$$

   and solve for b. This gives

   $$(1/2)b^{-1/2} - (3/2)b^{1/2} = 0,$$

   multiplying both sides by $-b^{1/2}$,

   $$(3/2)b - (1/2) = 0,$$

   and so $b = 1/3$.

   Therefore the maximum value must occur when $b = 1/3$. $a + b = 1$, so $a = 2/3$, and the maximum value is $(2/3)\sqrt{1/3}$.

2. **Suppose that**

$$g(x) = \frac{x^7}{7} - \frac{x^5}{5}.$$

a) How many relative minimum points are there of $g(x)$?

b) How many relative maximum points are there of $g(x)$?

c) How many inflection points are there of $g(x)$?

Differentiating

$$g'(x) = x^6 - x^4 = x^4(x^2 - 1)$$

and

$$g''(x) = 6x^5 - 4x^3 = x^3(6x^2 - 4).$$

Setting $g'(x) = 0$ and solving for x we get

$$x = 0,\ x = 1 \text{ or } x = -1.$$

Further

$$\begin{array}{lll} g'(x) > 0 & \text{if} & x < -1 \\ g'(x) < 0 & \text{if} & -1 < x < 0 \\ g'(x) < 0 & \text{if} & 0 < x < 1 \\ g'(x) > 0 & \text{if} & 1 < x \end{array}$$

Therefore

- $g(x)$ has a relative maximum at -1,
- $g(x)$ has neither a relative minimum or a relative maximum at $x = 0$,
- $g(x)$ has a relative minimum at 1.

Answer: There is $\boxed{1}$ point where $g(x)$ has a relative minimum.

Answer: There is $\boxed{1}$ point where $g(x)$ has a relative maximum.

To find the inflection points we set $g''(x) = 0$ and solve for x. We get

$$x = 0,\ x = \sqrt{2/3} \text{ or } x = -\sqrt{2/3}.$$

Further

$$\begin{array}{lll} g''(x) < 0 & \text{if} & x < -\sqrt{2/3} \\ g''(x) > 0 & \text{if} & -\sqrt{2/3} < x < 0 \\ g''(x) < 0 & \text{if} & 0 < x < \sqrt{2/3} \\ g''(x) > 0 & \text{if} & \sqrt{2/3} < x \end{array}$$

Therefore

- $g(x)$ has an inflection point at $x = -\sqrt{2/3}$,
- $g(x)$ has an inflection point at $x = 0$,
- $g(x)$ has an inflection point at $x = \sqrt{2/3}$.

Answer: There are $\boxed{3}$ inflection points.

3. **Find the points on the graph of $y = 2/x^2$ that are closest to the point $(0, 0)$.**

Suppose that (x, y) is a point on the graph of $y = 2/x^2$. The distance from (x, y) to $(0, 0)$ is:

$$D(x, y) = [(x - 0)^2 + (y - 0)^2]^{1/2} = [x^2 + y^2]^{1/2}.$$

Substituting $2/x^2$ for y we define

$$F(x) = [x^2 + (2/x^2)^2]^{1/2} = [x^2 + (4/x^4)]^{1/2}.$$

We must find the values (if any) of x at which $F(x)$ has a minimum value. Let

$$G(x) = (F(x))^2 = x^2 + (4/x^4).$$

Since the values of $F(x)$ are positive, $F(x)$ has a minimum value at $x = a$ if and only if $G(x)$ has a minimum value at $x = a$. Therefore we shall try to find the values of x at which the function $G(x)$ has an absolute minimum point.

The domain of $G(x)$ is

$$(-\infty, 0) \cup (0, \infty)$$

and so if $G(x)$ has a minimum value then it must occur at a point the derivative of $G(x)$ is 0. Differentiating

$$G'(x) = 2x - (16/x^5),$$

and

$$G''(x) = 2 + (80/x^6).$$

Therefore $G(x)$ is concave up everywhere on its domain.

Setting $G'(x) = 0$ gives the equation

$$2x = 16/x^5.$$

Multiplying both sides by x^5 and dividing both sides by 2 gives:

$$x^6 = 8.$$

Solving for x we get $x = \sqrt{2}$ or $x = -\sqrt{2}$.

$G(x)$ is concave up on $(0, \infty)$, therefore the value $G(\sqrt{2})$ must be the minimum value of $G(x)$ on $(0, \infty)$. Similarly $G(x)$ is concave up on $(-\infty, 0)$ and so $G(-\sqrt{2})$ is the minimum value of $G(x)$ on the interval $(-\infty, 0)$.

However

$$G(-\sqrt{2}) = G(\sqrt{2}) = 2 + 1 = 3.$$

Therefore $G(x)$ has a minimum value at both $x = -\sqrt{2}$ and at $x = \sqrt{2}$.

Thus the points

$$(-\sqrt{2}, 1) \text{ and } (\sqrt{2}, 1)$$

are points on the graph of $y = 2/x^2$ which are closest to $(0, 0)$ are the points. Further these are the only such points.

4. **Sketch the graph of**

$$f(x) = x^2/4 - x^{-2}/4.$$

First identify the domain. The domain of $f(x)$ is $(-\infty, 0) \cup (0, \infty)$.

Next determine where the function is positive, negative, and find the x-intercepts.

Solving $x^2/4 - x^{-2}/4 = 0$ gives $x^4 - 1 = 0$ (mutipling both sides by x^2). There are two solutions, 1 and -1.

Thus $f(x) > 0$ on $(-\infty, 1)$ and on $(1, \infty)$; and $f(x) < 0$ on $(-1, 0)$ and $(0, 1)$. The x-intercepts occur at $x = -1$ and at $x = 1$.

Next compute the derivative to look for relative extreme points and to identify where $f(x)$ is increasing and where it is decreasing.

$f'(x) = 2x/4 + 2x^{-3}/4$. Solving $f'(x) = 0$ gives $2x/4 + 2x^{-3}/4 = 0$ which reduces to $x^4 + 1 = 0$ (multiplying both sides by $4x^3$, and dividing both sides by 2). This has no solutions, so $f(x)$ has no relative extreme points.

Finally solve $f'(x) > 0$ to determine where $f(x)$ is increasing. But

$$f'(x) = 2x/4 + 2x^{-3}/4 = \frac{x^4 + 1}{2x^3}.$$

The numerator is always positive. So $f'(x) > 0$ on $(0, \infty)$ and $f'(x) < 0$ on $(-\infty, 0)$.

Summary: $f(x)$ has no relative extreme points, $f(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

Next compute the second derivative of $f(x)$ to look for inflection points and to identify where $f(x)$ is concave up and where $f(x)$ is concave down.

$f''(x) = 2/4 - 6x^{-4}/4$. Solving $f''(x) = 0$ gives $2 - 6x^{-4} = 0$ which reduces to $x^4 - 3 = 0$ (multiplying both sides by $2x^4$). This has two solutions, $-3^{1/4}$ and $3^{1/4}$.

Now solve $f''(x) > 0$ to determine where $f(x)$ is concave up.

$$f''(x) = 1/2 - 3x^{-4}/2 = \frac{x^4 - 3}{2x^4}.$$

The denominator is positive (for all $x \neq 0$). Thus solving $f''(x) > 0$ reduces to solving $x^4 - 3 > 0$. This gives $x < -3^{1/4}$ or $x > 3^{1/4}$.

Summary:

$f(x)$ is concave up on $\left(-\infty, -3^{1/4}\right)$;

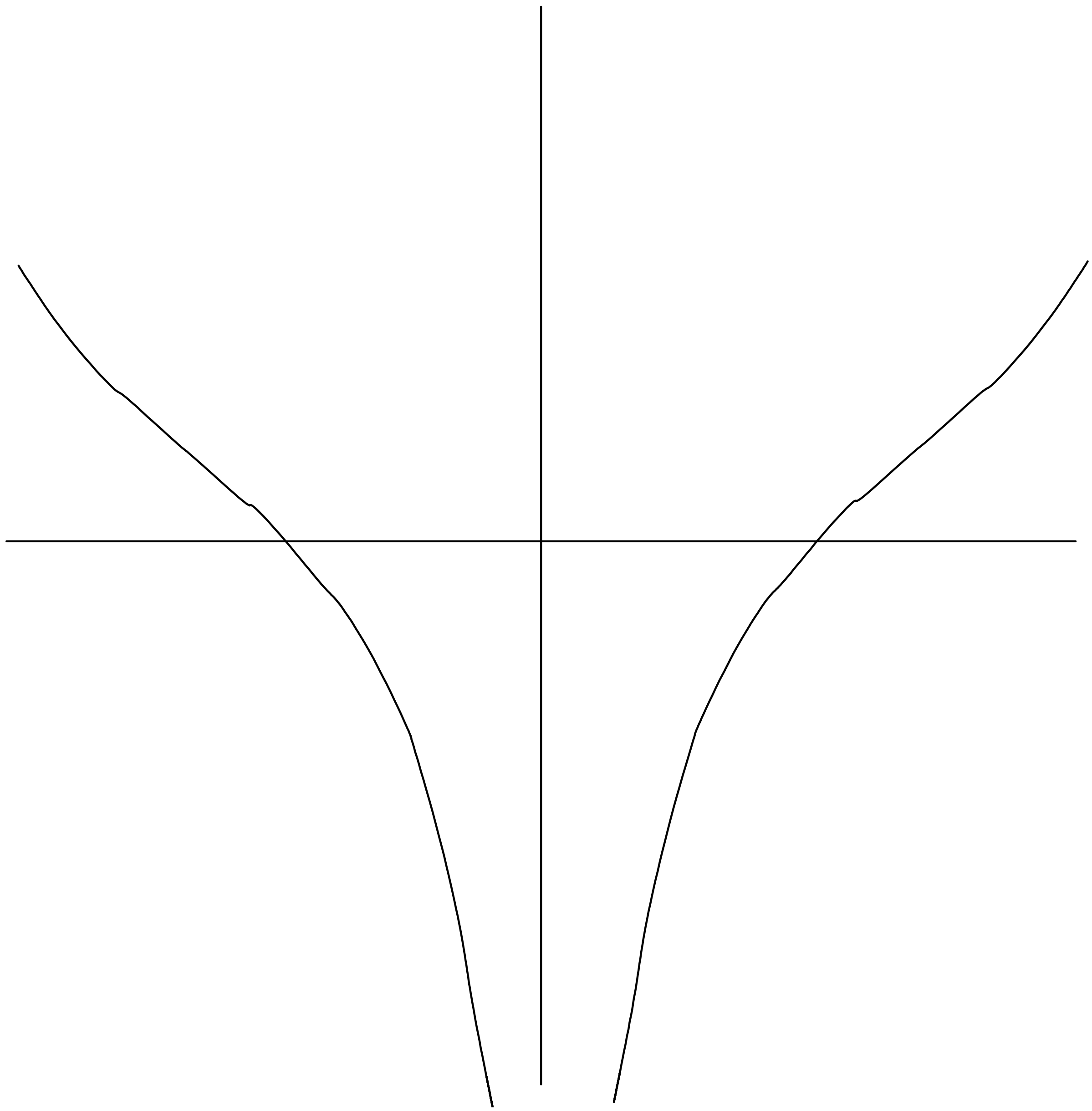
$f(x)$ is concave down on $\left(-3^{1/4}, 3^{1/4}\right)$;

$f(x)$ is concave up on $\left(3^{1/4}, \infty\right)$.

Thus $f(x)$ has inflection points at $x = -3^{1/4}$ and at $x = 3^{1/4}$. The two inflection points are,

$$(-3^{1/4}, 3^{1/2}/4 - 3^{-1/2}/4) \text{ and } (3^{1/4}, 3^{1/2}/4 - 3^{-1/2}/4)$$

Combining all this information we sketch the graph on the next page.



5. **Find the maximum value of**

$$f(x) = (x^8/8) - (x^6/6)$$

on the interval $[-2, 1]$

First solve $f'(x) = 0$. $f'(x) = x^7 - x^5 = x^5(x^2 - 1)$. So $f'(x) = 0$ gives $x^5(x^2 - 1) = 0$. This has three solutions, $-1, 0$, and 1.

The maximum value of $f(x)$ on the interval, $[-2, 1]$, must occur at -2, 1 or where $f'(x) = 0$.

Thus the maximum value of $f(x)$ on the interval, $[-2, 1]$ must occur at -2, -1, 0 or 1.

At this point one can simply evaluate $f(x)$ at each each of these 5 vlues of x and pick the largest.

Alternatively one can examine the graph of $f(x)$ more closely.

$f'(x) = x^5(x^2 - 1)$. Solving $f'(x) > 0$ gives $x^5(x^2 - 1) > 0$ which reduces to $-1 < x < 0$ or $x > 1$.

Thus

$f(x)$ is decreasing on $(-\infty, -1)$;
$f(x)$ is increasing on $(-1, 0)$;
$f(x)$ is decreasing on $(0, 1)$;
$f(x)$ is increasing on $(1, \infty)$.

Therefore $f(x)$ has relative minimum points at $x = -1$ and $x = 1$, and a relative maximum point at $x = 0$.

Thus the maximum value of $f(x)$ on the interval, $[-2, 1]$ cannot occur at -1 or 1 (because $f(x)$ has relative minimum points there). So the maximum value of $f(x)$ on the interval, $[-2, 1]$, must occur at either $-2, 0$, or 1.

Evaluating $f(x)$ at these vaules of x:

$f(-2) = (-2)^8/8 - (-2)^6/6 = 2^6(1/2 - 1/6) = 2^6/3$;
$f(0) = (0)^8/8 - (0)^6/6 = 0$;
$f(1) = (1)^8/8 - (1)^6/6 = 1/8 - 1/6 = -1/24$.

Clearly the largest of these values is $f(-2)$.

Answer: $\boxed{2^6/3}$

6. **Suppose** $y = ((\ln x)^2)^x$ **for** $x > 0$**. Find** $y'(1/e)$**.**

Recall that if $a > 0$ and $b > 0$ then $a^b = e^{b \ln a}$.

Thus

$$y = \left((\ln x)^2\right)^x = e^{x \ln((\ln x)^2)}$$

Now compute dy/dx using the chain rule:

$$dy/dx = e^{x \ln((\ln x)^2)} d[x \ln((\ln x)^2)]/dx$$

since $d[e^x]/dx = e^x$.

To compute $d[x \ln((\ln x)^2)]/dx$ use the product rule:

$$d[x \ln((\ln x)^2)]/dx = xd[\ln((\ln x)^2)]/dx + \ln((\ln x)^2)d[x]/dx$$

and so

$$d[x \ln((\ln x)^2)]/dx = xd[\ln((\ln x)^2)]/dx + \ln((\ln x)^2)$$

since $d[x]/dx = 1$.

Next using the chain rule again:

$$d[\ln((\ln x)^2)]/dx = (1/(\ln x)^2)d[(\ln x)^2]/dx$$

and using the general power rule:

$$d[(\ln x)^2]/dx = 2(\ln x)d[\ln x]/dx = 2(\ln x)/x.$$

Putting everything together:

$$dy/dx = e^{x \ln((\ln x)^2)} \left(x(1/(\ln x)^2)2(\ln x)/x + \ln((\ln x)^2)\right).$$

Evaluation at $1/e$:

$$y'(1/e) = e^0 \left((1/e)(-2e) + 0\right) = -2.$$

Answer $\boxed{\text{-2}}$

7. **Find the point, or points, on the graph of** $y = \sqrt{\ln x}$ **that are closest to the point** $(3/2, 0)$**.**

The domain of y is $[1, \infty)$. The distance between a point $(x, \sqrt{\ln x})$ on the graph of y and the point $(3/2, 0$ is:

Distance $= [(x-3/2)^2+(\sqrt{\ln x}-0)^2]^{1/2} = [(x-3/2)^2+(\sqrt{\ln x})^2]^{1/2} = [(x-3/2)^2+\ln x]^{1/2}$,

since $(\sqrt{\ln x})^2 = \ln x$ (why?).

Thus we must find the value(s) of x in $[1, \infty)$ where the function

$$f(x) = [(x-3/2)^2 + \ln x]^{1/2}$$

has global minimum points. These are the same values of x where the function

$$g(x) = (f(x))^2 = (x-3/2)^2 + \ln x$$

has global minimum points on $[1, \infty)$ (why?).

These values of x must be either 1 or where $g'(x) = 0$.

$$g'(x) = 2(x-3/2) + (1/x) = (2x-3) + (1/x) = \frac{x(2x-3)+1}{x}$$

and so (by factoring the numerator)

$$g'(x) = \frac{(2x-1)(x-1)}{x}.$$

Solving $g'(x) = 0$ we get $x = 1/2$ or $x = 1$.

Thus the global minimum point of $g(x)$ on $[1, \infty)$ must occur at $x = 1$. Check: Notice that $g'(x) > 0$ on $(1, \infty)$ and so $g(x)$ is increasing on $[1, \infty)$. This implies that $g(x)$ has a global minimum point on $[1, \infty)$ at $x = 1$.

Answer: $\boxed{(1, 0)}$

8. **Sketch the graph of**

$$f(x) = (x^4 - 1)/(x^4 + 1).$$

Clearly for all x, $f(x) = f(-x)$. Therefore the graph of $f(x)$ is symmetric about the y-axis.

The domain of $f(x)$ is $(-\infty, \infty)$. Further

$$\lim_{x\to\infty} f(x) = 1 = \lim_{x\to-\infty} f(x)$$

and so the line with equation, $y = 1$ is an horizontal asymptote for the graph of $f(x)$ as x tends to ∞ and as x tends to $-\infty$.

Differentiating using the quotient rule

$$f'(x) = \left([x^4+1](d(x^4-1)/dx) - [x^4-1](d(x^4+1)/dx)\right)/(x^4+1)^2.$$

Now

$$d(x^4-1)/dx = 4x^3 \text{ and } d(x^4+1)/dx = 4x^3,$$

and so

$$\begin{aligned} f'(x) &= ([x^4+1](4x^3) - [x^4-1](4x^3))/(x^4+1)^2 \\ &= ((4x^7+4x^3) - (4x^7-4x^3))/(x^4+1)^2 \\ &= 8x^3/(x^4+1)^2. \end{aligned}$$

Setting $f'(x) = 0$ and solving for x gives $x = 0$. Further

- $f'(x) < 0$ if $x < 0$,
- $f'(x) > 0$ if $x > 0$.

Therefore $f(x)$ has a minimum value at $x = 0$.

We now investigate concavity. Again differentiating using the quotient rule,

$$f''(x) = \left([x^4+1]^2(d(8x^3)/dx) - 8x^3(d(x^4+1)^2/dx)\right)/(x^4+1)^4.$$

Simplfying

$$\begin{aligned} f''(x) &= ([x^4+1]^2(24x^2) - (8x^3)[(2)(x^4+1)](4x^3))/(x^4+1)^4 \\ &= ([x^4+1][(x^4+1)(24x^2) - 64x^6])/(x^4+1)^4. \end{aligned}$$

Thus

$$f''(x) = \left((x^4+1)(24x^2) - 64x^6\right)/(x^4+1)^3$$

Finally

$$f''(x) = \left(24x^2 - 40x^6\right)/(x^4+1)^3 = \left([8x^2](3-5x^4)\right)/(x^4+1)^3.$$

We next look for inflection points. Setting $f''(x) = 0$ and solving for x we get three solutions

$$x = 0, x = (3/5)^{1/4} \text{ and } x = -(3/5)^{1/4}$$

Further

- $f''(x) < 0$ if $x^4 > 3/5$ i. e. if $x < -(3/5)^{1/4}$ or if $x > (3/5)^{1/4}$.
- $f''(x) > 0$ if $x^4 < 3/5$ i. e. if $-(3/5)^{1/4} < x < (3/5)^{1/4}$.

Therefore

- $f(x)$ is concave down if $x < -(3/5)^{1/4}$ or if $x > (3/5)^{1/4}$.
- $f(x)$ is concave up if $-(3/5)^{1/4} < x < (3/5)^{1/4}$.

Thus $f(x)$ has two inflection points; at $x = -(3/5)^{1/4}$ and at $x = (3/5)^{1/4}$.

9. **Find the maximum value of**

$$f(x) = \frac{x^2}{x^4 + 1}.$$

The domain of $f(x)$ is $(-\infty, \infty)$. Therefore if $f(x)$ has a maximum value it must occur at a point where the derivative is 0.

Differentiating using the quotient rule,

$$f'(x) = \left([x^4 + 1]d(x^2)/dx - x^2 d(x^4 + 1)/dx\right)/(x^4 + 1)^2.$$

Now

$$d(x^2)/dx = 2x$$

and

$$d(x^4 + 1)/dx = 4x^3.$$

Substituting and simplifying:

$$\begin{aligned}
f'(x) &= \left([x^4 + 1]d(x^2)/dx - x^2 d(x^4 + 1)/dx\right)/(x^4 + 1)^2 \\
&= \left(2x[x^4 + 1] - x^2[4x^3]\right)/(x^4 + 1)^2 \\
&= \left([2x^5 + 2x] - 4x^5\right)/(x^4 + 1)^2 \\
&= (2x - 2x^5)/(x^4 + 1)^2 \\
&= (2x)(1 - x^4)/(x^4 + 1)^2.
\end{aligned}$$

Setting $f'(x) = 0$ and solving for x we get three solutions,

$$x = -1, x = 0, x = 1.$$

Further

- $f'(x) > 0$ if $0 < x < 1$,
- $f'(x) < 0$ if $1 < x$.

Thus $f(x)$ has a relative maximum at $x = 1$ and further $f(1)$ is the maximum value of $f(x)$ on the interval $(0, \infty)$. For all x,

$$f(x) = f(-x)$$

and so the graph of $f(x)$ is symmetric around the x-axis. Now $f(0) = 0$ and $f(x) > 0$ if $x \neq 0$. Therefore the maximum value of $f(x)$ on $(0, \infty)$ must be the maximum value of $f(x)$.

Finally $f(1) = 1/2$ and so $1/2$ is the maximum value of $f(x)$.

10. **The curve defined by** $x^3 + y^3 = 9xy$ **is called the folium of Descartes.**

Let (a, b) **be the point where this curve intersects the graph of** $y = \sqrt{x}$ **<u>and</u>** $a > 0$**. Find the equation of the line tangent to this curve at the point** (a, b)**.**

First we find the point (a, b) where the graph of $y = \sqrt{x}$ intersects the curve defined by

$$x^3 + y^3 = 9xy.$$

Substituting $\sqrt{x}$ for y we get

$$x^3 + (\sqrt{x})^3 = 9x(\sqrt{x})$$

which we can rewrite as

$$x^3 + x^{3/2} = 9x^{3/2}.$$

Subtracting $x^{3/2}$ from both sides we get

$$x^3 = 8x^{3/2}.$$

Solving for x we get either $x = 0$ or

$$x^{3/2} = 8$$

which we obtain by dividing both sides by $x^{3/2}$.

The equation $x^{3/2} = 8$ has one solution, $x = 4$.

Thus the graph of $y = \sqrt{x}$ intersects the curve

$$x^3 + x^{3/2} = 9x^{3/2}$$

exactly twice, once where $x = 0$ and once where $x = 4$.

Therefore $a = 4$ and $b = \sqrt{4} = 2$.

Next we calculate dy/dx using implicit differentiation. Differentiating both sides of the equation

$$x^3 + y^3 = 9xy$$

we get

$$d(x^3 + y^3)/dx = d(9xy)/dx.$$

Now

$$\begin{aligned} d(x^3+y^3)/dx &= d(x^3)/dx + d(y^3)/dx \\ &= 3x^2 + 3y^2(dy/dx). \end{aligned}$$

and

$$d(9xy)/dx = 9y + 9x(dy/dx).$$

Substituting we get

$$3x^2 + 3y^2(dy/dx) = 9y + 9x(dy/dx),$$

which we can rewrite as

$$3x^2 - 9y = (9x - 3y^2)(dy/dx).$$

Solving for dy/dx we get that

$$dy/dx = (3x^2 - 9y)/(9x - 3y^2)$$

provided that $9x - 3y^2 \neq 0$.

We seek the slope of the line tangent at the point

$$(a, b) = (4, 2).$$

Now $9(4) - 3(2)^2 = 36 - 12 = 24 \neq 0$ and so we may use the formula to calculate the slope;

$$(3(4)^2 - 9(2))/(9(4) - 3(2)^2) = 30/24 = 5/4.$$

Finally we can use the point-slope formula to obtain the equation of the tangent line.

$$\frac{y-2}{x-4} = 5/4$$

which simplifies to give

$$y - 2 = (5/4)x - 5$$

or simply

$$y = (5/4)x - 3.$$

11. **Suppose x and y are differentiable functions of t related for all t by the equation**

$$y^2 \ln x + 2y + 1 = 0.$$

Suppose that $x(0) = e$. Find $x'(0)$.

Differentiating both sides of the equation

$$y^2 \ln x + 2y + 1 = 0$$

with respect to t gives

$$\frac{(y^2 \ln x + 2y + 1)}{dt} = \frac{d(0)}{dt}.$$

Now

$$\frac{d(0)}{dt} = 0$$

and

$$\begin{aligned} d(y^2 \ln x + 2y + 1)/dt &= d(y^2 \ln x)/dt + d(2y)/dt + d(1)/dt \\ &= 2y(dy/dt)(\ln x) + y^2(1/x)(dx/dt) + 2(dy/dt) \end{aligned}$$

Thus for all values of t:

$$2y(\ln x)\frac{dy}{dt} + y^2(1/x)\frac{dx}{dt} + 2\frac{dy}{dt} = 0.$$

When

$$x = e$$

the equation

$$y^2 \ln x + 2y + 1 = 0$$

gives that $y^2 + 2y + 1 = 0$, so $y = -1$.

Now $x(0) = e$, and so the equation

$$2y(\ln x)\frac{dy}{dt} + y^2(1/x)\frac{dx}{dt} + 2\frac{dy}{dt} = 0$$

gives

$$2(-1)(\ln e)y'(0) + (-1)^2(1/e)x'(0) + 2y'(0) = 0;$$

and so

$$(-2)y'(0) + (1/e)x'(0) + 2y'(0) = 0.$$

This simplifies to:

$$(1/e)x'(0) = 0$$

and so $x'(0) = 0$.

12. **Suppose** $f(x) = 2x^2 \ln x - x^2 + 4x$

How many relative extreme points does $f(x)$ **have? Why?**

First calculate the derivative of $f(x)$:

$$\frac{d[f(x)]}{dx} = \frac{d[2x^2 \ln x]}{dx} - \frac{d[x^2]}{dx} + \frac{d[4x]}{dx}$$

and so

$$\frac{d[f(x)]}{dx} = \frac{d[2x^2 \ln x]}{dx} - 2x + 4.$$

Using the product rule:

$$\frac{d[2x^2 \ln x]}{dx} = 2x^2 \frac{d[\ln x]}{dx} + 2 \ln x \frac{d[x^2]}{dx}$$

and so

$$\frac{d[2x^2 \ln x]}{dx} = 2x^2(1/x) + 2 \ln x(2x) = 2x + 4x \ln x.$$

Substituting:

$$\frac{d[f(x)]}{dx} = \frac{d[2x^2 \ln x]}{dx} - 2x = 4x \ln x + 4.$$

Solving $4x \ln x + 4 = 0$ looks difficult. The hint suggests looking for the minimum value of $g(x) = f'(x)$.

The domain of $g(x)$ is $(0, \infty)$ so the minimum value must occur where either $g'(x) = 0$ or where $g'(x)$ is undefined.

Using the product rule:

$$g'(x) = 4\frac{d[x \ln x]}{dx} = 4(x\frac{d[\ln x]}{dx} + (\ln x)\frac{d[x]}{dx}$$

and so

$$g'(x) = 4(1 + \ln x).$$

Thus $g'(x)$ is defined for all $x > 0$ and so the minimum vaule of $g(x)$ must occur where $g'(x) = 0$.

Solving $g'(x) = 0$ gives $x = 1/e$. Further $g'(x) < 0$ for $0 < x < 1/e$ and $g'(x) > 0$ for $1/e < x$.

Therefore $g(x)$ has a global minimum point at $x = 1/e$.

Finally, $g(1/e) = 4((1/e)\ln(1/e) + 1) = 4((1/e)(-1) + 1) = 4(1 - 1/e)$. Since $e > 1$, $1 - 1/e > 0$ and so $g(x) > 0$ for all x in $(\infty, 0)$.

But $g(x) = f'(x)$ and so $f(x)$ has no relative extreme points since $f'(x) \neq 0$ for all x in $(0, \infty)$.

13. **Find the equation of the line tangent to the ellipse**

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

at the point where the ellipse intersects the graph of $y = (3/2)\sqrt{3x}$**.**

We first solve for the point of intersection. Substituting $(3/2)\sqrt{3x}$ for y in the equation for the ellipse we get:

$$\frac{x^2}{4} + \frac{[(3/2)\sqrt{3x}]^2}{9} = 1$$

which simplifies to:

$$\frac{x^2}{4} + \frac{[(9/4)(3x)}{9} = 1.$$

This further reduces to:

$$x^2 + 3x = 4.$$

There are two solutions, $x = -4$ and $x = 1$. But $x > 0$ since we are looking for a point on the graph of $y = (3/2)\sqrt{3x}$, so $x = 1$. This gives $y = (3/2)\sqrt{3}$.

Next we compute the slope of the line tangent to the ellipse at the point $(1, (3/2)\sqrt{3})$, using implicit differentiation.

Differentiating both sides of the equation,

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

with respect to x we get,

$$x/2 + (2/9)y\frac{dy}{dx} = 0$$

and solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -(9/4)(x/y).$$

Thus at the point, $(1, (3/2)\sqrt{3})$, the slope of the tangent line is:

$$-(9/4)(1/[3/2\sqrt{3}]) = -(3/2)/\sqrt{3} = -(1/2)\sqrt{3}.$$

Finaly the equation for the tamgent line is:

$$y - (3/2)\sqrt{3} = (-(1/2)\sqrt{3})(x - 1)$$

or

$$y = 2\sqrt{3} - (\sqrt{3}/2)x.$$

14. Suppose $g(x)$ is a differentiable function with domain $(-\infty, \infty)$ such that for all x, $g'(x) = 4g(x)$. Suppose $g(1) = 1$. Find $g^{(2)}(2)$.

    By the theorem in the book, there exists a constant C such that

    $$g(x) = Ce^{4x}.$$

    We need to solve for C. We are given,

    $$g(1) = 1.$$

    Therefore $Ce^{4\cdot 1} = 1$. This gives

    $$C = e^{-4}$$

    and so $g(x) = e^{-4}e^{4x}$.

    Now we can compute $g^2(x)$ by differentiating twice, getting

    $$g^{(2)}(x) = e^{-4}(4)(4)e^{4x} = 16e^{-4}e^{4x}.$$

    Evaluating at $x = 2$,

    $$g^{(2)}(2) = 16e^{-4}e^{4\cdot 2} = 16e^{-4}e^{8} = 16e^{4}.$$

    Answer: $\boxed{g^{(2)}(2) = 16e^4}$

15. **Suppose that**

$$g(x) = |x| + x^3.$$

a) **How many relative minimum points are there of** $g(x)$**?**

b) **How many relative maximum points are there of** $g(x)$**?**

c) **How many inflection points are there of** $g(x)$**?**

$$\begin{array}{lll} g(x) = x + x^3 & \text{if} & x > 0 \\ g(x) = 0 & \text{if} & x = 0 \\ g(x) = x^3 - x & \text{if} & x < 0 \end{array}$$

Differentiating

$$\begin{array}{lll} g'(x) = 1 + 3x^2 & \text{if} & x > 0 \\ g'(x) = 3x^2 - 1 & \text{if} & x < 0 \end{array}$$

By matching, $g(x)$ is not differentiable at $x = 0$.

Solving $g'(x) = 0$ gives $x = -1/\sqrt{3}$. Thus the only candidates for relative extreme points for $g(x)$ are at $x = -1/\sqrt{3}$ and at $x = 0$.

Determining the sign of $g'(x)$ yields:

$$\begin{array}{lll} g'(x) > 0 & \text{if} & x < -1/\sqrt{3} \\ g'(x) < 0 & \text{if} & -1/\sqrt{3} < x < 0 \\ g'(x) > 0 & \text{if} & 0 < x \end{array}$$

Therefore

- $g(x)$ is increasing on $(-\infty, -1/\sqrt{3}]$,
- $g(x)$ is decreasing on $[-1/\sqrt{3}, 0]$,
- $g(x)$ is increasing on $[0, -\infty, \infty)$.

Therefore

- $g(x)$ has a relative maximum point at $-1/\sqrt{3}$,
- $g(x)$ has relative minimum point at $x = 0$.

Answer for part (a): There is $\boxed{1}$ relative minimum point.

Answer for part (b): There is $\boxed{1}$ relative maximum point.

To find the inflection points we must compute first compute $g''(x)$.

$$\begin{array}{l} g''(x) = 6x \quad \text{if} \quad x > 0 \\ g''(x) = 6x \quad \text{if} \quad x < 0 \end{array}$$

and $g''(0)$ is not defined (why?).

Therefore

- $g(x)$ is concave down on $(-\infty, 0)$
- $g(x)$ is concave up on $(0, \infty)$.

$g(x)$ is continuous at $x = 0$ and so $g(x)$ has an inflection point at $x = 0$ and this is the only inflection point of $g(x)$.

Answer for part (c): There is $\boxed{1}$ inflection point.