1. Differentiate $f(x) = e^{\frac{x-1}{x^2+1}}$.

$$f'(x) = e^{\frac{x-1}{x^2+1}} \cdot \left(\frac{x-1}{x^2+1}\right)'$$

$$= e^{\frac{x-1}{x^2+1}} \cdot \frac{(x-1)'(x^2+1) - (x-1)(x^2+1)'}{(x^2+1)^2}$$

$$= e^{\frac{x-1}{x^2+1}} \cdot \frac{x^2+1 - (x-1)\cdot 2x}{(x^2+1)^2}$$

$$= e^{\frac{x-1}{x^2+1}} \cdot \frac{x^2+1-2x^2+2x}{(x^2+1)^2}$$

$$= \boxed{e^{\frac{x-1}{x^2+1}} \cdot \frac{-x^2+2x+1}{(x^2+1)^2}}$$

2. Find the equation of the tangent lines to the graph of $x^2y^4 = 1$ at the point $(4, \frac{1}{2})$ and at the point $(4, -\frac{1}{2})$.

$$x^2y^4 = 1$$

$$\frac{d}{dx} x^2y^4 = \frac{d}{dx} 1$$

$$2xy^4 + x^2 \cdot 4y^3 \cdot \frac{dy}{dx} = 0$$

$$4x^2y^3 \frac{dy}{dx} = -2xy^4$$

$$\frac{dy}{dx} = \frac{-2xy^4}{4x^2y^3}$$

$$= \frac{-y}{2x}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=4 \\ y=1/2}} = \frac{-1/2}{2 \cdot 4} = \frac{-1}{16}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=4 \\ y=-1/2}} = \frac{-(-1/2)}{2 \cdot 4} = \frac{1}{16}$$

The equation of the tangent line at $(4, 1/2)$ is $y - 1/2 = -1/16(x-4)$

$$\boxed{y = -1/16\, x + 3/4}$$

The equation of the tangent line at $(4, -1/2)$ is

$$y - (-1/2) = 1/16(x-4)$$

$$\boxed{y = \frac{1}{16}x - 3/4}$$

3. Farmer Brown has 40 feet of fencing and wishes to make a rectangular fenced-in area for his flock of chickens. If he uses his house for one side of the fence, what is the maximum area he can enclose?

Objective

Maximize Area

Objective Equation

Area = length * width

Let A = area
l = length
w = width

$A = lw$

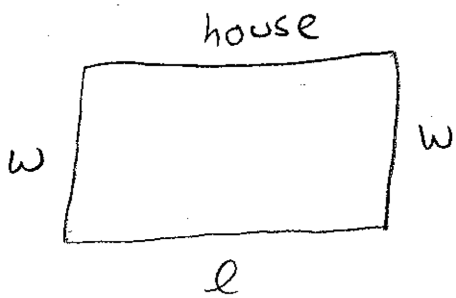
Constraints

$2w + l = 40$

$\Rightarrow l = 40 - 2w$

Objective Function

$A(w) = (40-2w) \cdot w$
$= 40w - 2w^2$



Maximize
$A(w) = 40w - 2w^2$

$A'(w) = 40 - 4w$
$0 = 40 - 4w$
$4w = 40$
$w = 10$

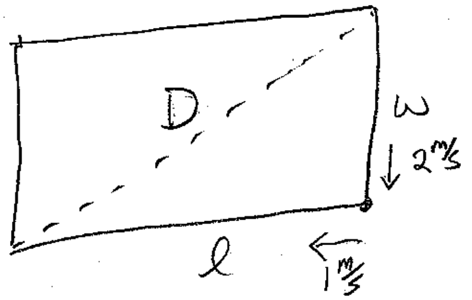
Since $A(w)$ is quadratic with negative leading coefficient, this gives a max.

When $w = 10$,

The maximum area is

$A(10) = (40 - 2(10)) \cdot 10 = 20 \cdot 10$
$= \boxed{200 \text{ square feet}}$

4. The length ℓ of a rectangle is decreasing at a rate of 1 cm/sec and the width w of the rectangle is increasing at a rate of 2 cm/sec. Find the rates of change for the area and for the length of the diagonal when $\ell = 5$ and $w = 12$. Indicate whether these quantities are decreasing or increasing.



Let A = area = $\ell \cdot w$

Let D = diagonal length = $\sqrt{\ell^2 + w^2}$

We are given that

$$\frac{d\ell}{dt} = -1 \quad \text{and} \quad \frac{dw}{dt} = 2$$

and we are asked to find

$$\left.\frac{dA}{dt}\right|_{\substack{\ell=5\\ w=12}} \quad \text{and} \quad \left.\frac{dD}{dt}\right|_{\substack{\ell=5\\ w=12}}$$

$$\frac{dA}{dt} = \frac{d}{dt}[\ell \cdot w] = \frac{d\ell}{dt} \cdot w + \ell \cdot \frac{dw}{dt}$$

$$\left.\frac{dA}{dt}\right|_{\substack{\ell=5\\ w=12}} = (-1)(12) + (5)(2) = \boxed{-2 \text{ cm}^2/\text{sec}}$$

$$\frac{dD}{dt} = \frac{d}{dt}\left[\sqrt{\ell^2+w^2}\right] = \frac{1}{2\sqrt{\ell^2+w^2}}\left[2\ell\frac{d\ell}{dt} + 2w\frac{dw}{dt}\right] = \frac{\ell\frac{d\ell}{dt} + w\frac{dw}{dt}}{\sqrt{\ell^2+w^2}}$$

$$\left.\frac{dD}{dt}\right|_{\substack{\ell=5\\ w=12}} = \frac{(5)(-1) + (12)(2)}{\sqrt{12^2 + 5^2}} = \frac{19}{\sqrt{169}} = \boxed{\frac{19}{13} \frac{\text{cm}}{\text{sec}}}$$

5. (15 points) Let $f(x) = e^{-x^2}$. Find all x-intercepts, asymptotes, relative extreme points, and points of inflection for $y = f(x)$. State how you know that the points you find are correctly identified.

$f(x) = e^{-x^2}$

Domain: $(-\infty, \infty)$

$\lim_{x \to \pm\infty} f(x) = 0$

so horizontal asymptote at $y = 0$

Set $f(x) = 0$

$e^{-x^2} = 0$

No Solution

So f has no x-intercepts

Furthermore $f(x) > 0$ for every x.

$f'(x) = e^{-x^2} \cdot -2x$

$= -2xe^{-x^2}$

$0 = -2xe^{-x^2}$

$0 = x$

$f(x)$ is $\begin{cases} > 0 & \text{if } x < 0 \\ < 0 & \text{if } x > 0 \end{cases}$

So f is increasing on $(-\infty, 0)$
decreasing on $(0, \infty)$

f has a local and absolute max at $x = 0$.

$f''(x) = -2e^{-x^2} + (-2x)(e^{-x^2})(-2x)$

$= e^{-x^2}[-2 + 4x^2]$

$0 = e^{-x^2}[-2 + 4x^2]$

$0 = -2 + 4x^2$

$2 = 4x^2$

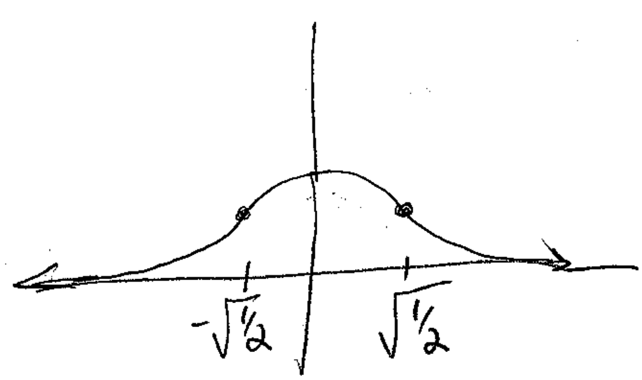
$1/2 = x^2$

$\pm\sqrt{1/2} = x$

$f(x)$ is $\begin{cases} > 0 & \text{if } x < -\sqrt{1/2} \\ < 0 & \text{if } -\sqrt{1/2} < x < \sqrt{1/2} \\ > 0 & \text{if } x > \sqrt{1/2} \end{cases}$

f is concave up on $(-\infty, -\sqrt{1/2})$ and on $(\sqrt{1/2}, \infty)$

f is concave down on $(-\sqrt{1/2}, \sqrt{1/2})$



6. (15 points) Sketch the graph of $y = x^4 - 4x^3$. Clearly indicate all x-intercepts, relative and absolute extreme points, and points of inflection.

$y = x^4 - 4x^3$

Domain: $(-\infty, \infty)$

Set $y = 0$

$x^4 - 4x^3 = 0$

$x^3(x-4) = 0$

$x = 0$ or $x = 4$

$$f(x) \text{ is } \begin{cases} >0 & \text{if } x<0 \\ <0 & \text{if } 0<x<4 \\ >0 & \text{if } x>4 \end{cases}$$

$y' = 4x^3 - 12x^2$

set $y' = 0$

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0$ or $x = 3$

$$f'(x) \text{ is } \begin{cases} <0 & \text{if } x<0 \\ <0 & \text{if } 0<x<3 \\ >0 & \text{if } x>3 \end{cases}$$

So f is increasing on $(-\infty, 3)$
decreasing on $(3, \infty)$

f has a local and absolute minimum at $x = 3$.

$y'' = 12x^2 - 24x$

set $y'' = 0$

$12x^2 - 24x = 0$

$x(12x - 24) = 0$

$x = 0$ or $x = 2$

$$f''(x) \text{ is } \begin{cases} >0 & \text{if } x<0 \\ <0 & \text{if } 0<x<2 \\ >0 & \text{if } x>2 \end{cases}$$

so f is concave up on $(-\infty, 0)$ and $(2, \infty)$

f is concave down on $(0, 2)$

f has inflection points at $x=0$ and $x=2$