Midterm # 2, Math 16A, Fall 2012
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YOUR NAME:

YOUR GSI's NAME:

YOUR SECTION NUMBER:

This exam contains 8 problems. Each of these problems is worth 10 points for a total of 80 points. You must **circle your answer for each problem** and **show your work** to get credit. Important: Do all work on the exam, use the back if you need more space.

No electronic devices, notes, or books

Please do not write below here.

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1. Find the slope of the line tangent to the circle

$$x^2 + y^2 = 5$$

at the point where the circle intersects the graph of $y = 2\sqrt{x}$.

Solution:

To find the implicit derivative (for finding tangent slopes) on the curve $x^2+y^2 = 5$ $d/dx = '$ to both sides of $x^2 + y^2 = 5$ to obtain

$$2x + 2yy' = 0$$

$$y' = -2x/(2y) = -x/y.$$

To find the point of intersection, substitute $y = 2\sqrt{(}x)$ in $x^2 + y^2 = 5$ to obtain

$$x^2 + 4x = 5$$

$$(x - 1)(x + 5) = 0$$

$$x = 1 \text{ or } x = -5$$

Since $y = 2\sqrt{x}$ we must have $x \geq 0$, so $x = 1, y = 2$ is the point. At that point $y' = -x/y = -1/2$.

2. Suppose that

$$h(x) = -|x| + x^5$$

   (a) Does $h(x)$ have any relative maximum or relative minimum points? If so find and identify them.

   (b) Does $h(x)$ have any inflection points? If so find them.

$$h(x) = \begin{cases} -x + x^5 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x + x^5 & \text{if } x < 0 \end{cases}$$

Differentiating

$$h'(x) = \begin{cases} -1 + 5x^4 & \text{if } x > 0 \\ 1 + 5x^4 & \text{if } x < 0 \end{cases}$$

By matching, $h(x)$ is not differentiable at $x = 0$.

Solving $h(x) = 0$ gives $x = \sqrt[1/4]{1/5}$. Thus the only candidates for relative extreme points for h(x) are at $x = \sqrt[1/4]{1/5}$ and at $x = 0$.

Determining the sign of $h'(x)$ yields:

$$\begin{cases} h'(x) > 0 & \text{if } x > \sqrt[1/4]{1/5} \\ h'(x) < 0 & \text{if } 0 < x < \sqrt[1/4]{1/5} \\ h'(x) > 0 & \text{if } x < 0 \end{cases}$$

Therefore

$$h(x)\text{is} \begin{cases} \text{increasing} & \text{on } (-\infty, 0] \\ \text{decreasing} & \text{on } [0, \sqrt[1/4]{1/5}] \\ \text{increasing} & \text{on } [\sqrt[1/4]{1/5}, \infty) \end{cases}$$

Thus $h(x)$ has a relative maximum point at $x = 0$ and a relative minimum point at $x = \sqrt[1/4]{1/5}$.

Answer for part (a): There is one relative maximum point which is $(0, 0)$. There is one relative minimum point which is $(\sqrt[1/4]{1/5}, -(4/5)(\sqrt[1/4]{1/5}))$.

To find the inflection points we compute $h''(x)$.

$$h''(x) = 20x^3 \text{ if } x > 0 \text{ or } x < 0$$

$h''(0)$ is undefined since $h'(0)$ is undefined.

Thus $h(x)$ is concave down in $(-\infty, 0)$ and concave up in $(0, \infty)$.

$h(x)$ is continuous at $x = 0$ and so $h(x)$ has an inflection point at $x = 0$ and this is the only inflection point of $h(x)$.

Answer for part (b): There is one inflection point which is $(0, 0)$.

3. Find the minimum value of $2a\ln(a) + a$ where $a > 0$.

Solution: We take the derivative of the function $f(a) = 2a\ln(a) + a$ using the product rule and set it equal to zero.

$$f'(a) = 2\ln(a) + 2a \cdot \frac{1}{a} + 1 = 2\ln(a) + 3 = 0$$

Now we solve for a:

$$\ln(a) = -\frac{3}{2}$$

$$e^{\ln(a)} = e^{-\frac{3}{2}}$$

Since the logarithm function and the exponential function are inverses, our only critical point is

$$a = e^{-\frac{3}{2}}$$

Check that this is a relative minimum by using the second derivative test:

$$f''(a) = \frac{2}{a}$$

$$f''(e^{-\frac{3}{2}}) = 2e^{\frac{3}{2}} > 0$$

Since there is no other critical points in the domain $a > 0$, this relative minimum must be the absolute minimum, so the minimum value is

$$f(e^{-\frac{3}{2}}) = 2e^{-\frac{3}{2}} \cdot \ln(e^{-\frac{3}{2}}) + e^{-\frac{3}{2}} = -2e^{-\frac{3}{2}}$$

4. Consider the curve defined by the equation $x^4 + y^4 = 8xy$. Find the equation of the line tangent to the curve at the point $(2, 2)$.

Solution: First we do implicit differentiation. Use the product rule for the right hand side:

$$4x^3 + 4y^3y' = 8xy' + 8y$$

Now we solve for y':

$$y' = \frac{8y - 4x^3}{4y^3 - 8x} = \frac{2y - x^3}{y^3 - 2x}$$

Plug in the point $(2, 2)$ to find the slope.

$$m = \frac{2 \cdot 2 - 2^3}{2^3 - 2 \cdot 2} = \frac{-4}{4} = -1$$

To find the equation of the tangent line, we use the point slope formula:

$$y - 2 = -1 \cdot (x - 2)$$

So the equation of the tangent line is

$$y = -x + 4$$

5. Suppose

$$f(x) = e^x - \frac{x^3}{6} + x^2 + 3x + 4$$

(a) Let $g(x) = f^{(2)}(x)$. Find the minimum value of $g(x)$.

(b) Does $f(x)$ have an inflection point? Why?

Solution:

(a) First find $g(x)$:

$$f'(x) = e^x - \frac{3x^2}{6} + 2x + 3$$

$$g(x) = f''(x) = e^x - x + 2$$

If $g(x)$ has a minimum, it will occur at a critical point. Solve

$$g'(x) = e^x - 1 = 0$$

giving us $x = \ln 1 = 0$.
Is this a minimum? One way to check is by second derivative test:

$$g''(x) = e^x; \quad g''(0) = 1 > 0$$

So $g(x)$ has derivative equal to 0, and is concave up, at $x = 0$, so $g(x)$ has a minimum there. The minimum value of $g(x)$ is

$$g(0) = e^0 - 0 + 2 = 3$$

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(b) No, $f(x)$ does not have an inflection point.
An inflection point occurs when $f(x)$ changes concavity, i.e. $f''(x)$ changes sign. However, we know from part (a) that $f''(x)$ has a minimum value of 3, which is positive. This tells us that $f''(x)$ is always positive, so $f(x)$ never changes concavity.

6. Find the shortest possible distance from a point on the graph of $y = x^2$ to the point $(0, 1)$.

Solution:

The distance formula from a point (x, y) to the point $(0, 1)$ is given by

$$d(x,y) = \sqrt{(x-0)^2 + (y-1)^2}$$

Plugging in $y = x^2$ we have

$$d = \sqrt{x^4 - x^2 + 1} \tag{1}$$

We want to minimize the function d. The $x-$coordinate of the minimum of the functions d and d^2 are the same. So we minimize $f(x) = d^2 = x^4 - x^2 + 1$. Setting $f'(x) = 0$ and solving for x we get

$$f'(x) = 4x^3 - 2x = 2x(2x^2 - 1) = 0 \qquad \Rightarrow \qquad x = 0 \text{ or } x = \pm\frac{1}{\sqrt{2}}$$

Analyzing the sign of f' we conclude $f'(x) = \begin{cases} < 0 & \text{if } x < \frac{-1}{\sqrt{2}}, \\ > 0 & \text{if } \frac{-1}{\sqrt{2}} < x < 0, \\ < 0 & \text{if } 0 < x < \frac{1}{\sqrt{2}}, \\ > 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$. Therefore, we conclude that $x = \pm\frac{1}{\sqrt{2}}$ are local minimum (you could also use the second derivative test to conclude this).

If we use formula (**??**) to compute the distance for $x = \pm\frac{1}{\sqrt{2}}$, we get the same answer (by the symmetry of the problem you should already suspect this) and conclude that the minimum distance is $d = \sqrt{3}/2$.

7. Find the maximum value of $a \cdot b$ given that $3a + b = 6$.

   Solution: We want to maximize $P = ab$ so we write it as a function of only one variable. $3a + b = 6$ therefore $b = 6 - 3a$ and $P = ab = a(6 - 3a) = 6a - 3a^2$.

   $P' = 6 - 6a = 0$ hence $a = 1$ is the only critical point of P and because $P'' = -6$ is negative it is a maximum.

   When $a = 1$ we have $b = 6 - 3 = 3$ and, $ab = 3$ is the maximum value.

8. Suppose $f(x)$ is a function with domain $(-\infty, \infty)$ such that for all x, $f'(x)$ exists and $f'(x) = 2f(x)$.

   Suppose $f^{(2)}(2) = 1$. Find $f^{(3)}(3)$.

   Solution: We have that $f(x) = Ce^{2x}$, for some constant C, by the theorem in class on the uniqueness of solutions to differential equations [1].

   This makes $f^{(2)}(2) = 2^2Ce^{2\cdot 2} = 4Ce^4$; we also know, by stipulation, that $f^{(2)}(2) = 1$; accordingly, $C = \frac{1}{4}e^{-4}$. Finally, we can compute $f^{(3)}(3) = 2^3Ce^{2\cdot 3} = \frac{8}{4}e^{-4}e^6 = 2e^2$.

[1]For those who would like to see this in more detail, consider the function $f(x)e^{-2x}$. Its derivative, by the product rule, is $2f(x)e^{-2x} - 2f(x)e^{-2x} = 0$; thus, it is some constant C. From this we can conclude, by multiplying through by e^{2x}, that $f(x) = Ce^{2x}$.