Midterm #1, Math 16a, Fall 2010

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1. Find

$$\lim_{x\to\infty} \frac{x^2 - 4x + 4}{2x^2 - 9}$$

solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2 - 4x + 4}{2x^2 - 9} &= \lim_{x\to\infty} \frac{\left(\frac{x^2-4x+4}{x^2}\right)}{\left(\frac{2x^2-9}{x^2}\right)} \\
&= \lim_{x\to\infty} \frac{\left(1 - \frac{4}{x} + \frac{4}{x^2}\right)}{\left(2 - \frac{9}{x^2}\right)} \\
&= \frac{1 - 0 + 0}{2 - 0} \\
&= \frac{1}{2}
\end{aligned}$$

2. Use the definition of the derivative as a limit to show that $f(x) = x^{1/3}$ is not differentiable at $x = 0$.

solution: Using the definition of the derivative at $x = 0$, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0} \frac{(0+h)^{\frac{1}{3}} - 0^{\frac{1}{3}}}{h} \\
&= \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h} \\
&= \lim_{h\to 0} h^{-\frac{2}{3}} \\
& D.N.E. \text{(does not exist)}
\end{aligned}$$

Since the limit defining the derivative does not exist, the function is not differentiable at $x = 0$.

3. Use the definition of the derivative as a limit to show that $f(x) = x^{1/3}$ is differentiable at $x = 2$.

   HINT: $(p - q) = (p^{1/3} - q^{1/3})((p^{2/3} + p^{1/3}q^{1/3} + q^{2/3})$

   solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{(2+h)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{h} \\
&= \lim_{h\to 0} \frac{\left((2+h)^{\frac{1}{3}} - 2^{\frac{1}{3}}\right) \cdot \left[(2+h)^{\frac{2}{3}} + (2+h)^{\frac{1}{3}} 2^{\frac{1}{3}} + 2^{\frac{2}{3}}\right]}{h \cdot \left[(2+h)^{\frac{2}{3}} + (2+h)^{\frac{1}{3}} 2^{\frac{1}{3}} + 2^{\frac{2}{3}}\right]} \\
&= \lim_{h\to 0} \frac{(2+h) - 2}{h \cdot \left[(2+h)^{\frac{2}{3}} + (2+h)^{\frac{1}{3}} 2^{\frac{1}{3}} + 2^{\frac{2}{3}}\right]} \\
&= \lim_{h\to 0} \frac{h}{h \cdot \left[(2+h)^{\frac{2}{3}} + (2+h)^{\frac{1}{3}} 2^{\frac{1}{3}} + 2^{\frac{2}{3}}\right]} \\
&= \lim_{h\to 0} \frac{1}{\left[(2+h)^{\frac{2}{3}} + (2+h)^{\frac{1}{3}} 2^{\frac{1}{3}} + 2^{\frac{2}{3}}\right]} \\
&= \frac{1}{2^{\frac{2}{3}} + 2^{\frac{1}{3}} 2^{\frac{1}{3}} + 2^{\frac{2}{3}}}
\end{aligned}$$

   Since the limit exists, the function is differentiable at $x = 2$.

4. Find the derivative of

$$y = x + 1 + \sqrt{x+1}$$

   solution:

$$\begin{aligned}
y &= x + 1 + \sqrt{x+1} \\
&= x + 1 + (x+1)^{\frac{1}{2}} \\
y' &= 1 + 0 + \frac{1}{2}(x+1)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(x+1) \\
&= 1 + \frac{1}{2}(x+1)^{-\frac{1}{2}}
\end{aligned}$$

5. Find the equation of the line tangent to the curve

$$y = \frac{1}{x^3 - x}$$

at $x = 2$.

solution: First we compute the slope of the tangent line, which is $y'(2)$

$$\begin{aligned}
y &= \frac{1}{x^3 - x} \\
&= (x^3 - x)^{-1} \\
y' &= -1 \cdot (x^3 - x)^{-1-1} \cdot \frac{d}{dx}(x^3 - x) \\
&= -1(x^3 - x)^{-2} \cdot (3x^2 - 1) \\
y'(2) &= -1(2^3 - 2)^{-2} \cdot (3 \cdot (2)^2 - 1) \\
&= -1 \cdot 6^{-2} \cdot 11 \\
&= \frac{-11}{36}
\end{aligned}$$

Then we compute the relevant point on the curve: $(2, y(2))$.

$$\begin{aligned}
y &= \frac{1}{x^3 - x} \\
y(2) &= \frac{1}{2^3 - 2} \\
&= \frac{1}{6}
\end{aligned}$$

So the point is $(2, \frac{1}{6})$. We can now write the point slope equation of the line:

$$
\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - \frac{1}{6} &= -\frac{11}{36}(x - 2) \\
y &= -\frac{11}{36}x + \frac{11}{18} + \frac{1}{6} \\
y &= -\frac{11}{36}x + \frac{14}{18} \\
y &= -\frac{11}{36}x + \frac{7}{9}
\end{aligned}
$$

6. Using the derivative, find an approximate value of $126^{1/3}$.

   solution: Since $125^{\frac{1}{3}} = 5$, we use the approximation $f(a + h) \approx f(a) + hf'(a)$ using $f(x) = x^{\frac{1}{3}}$, $a = 125$ and $h = 1$.

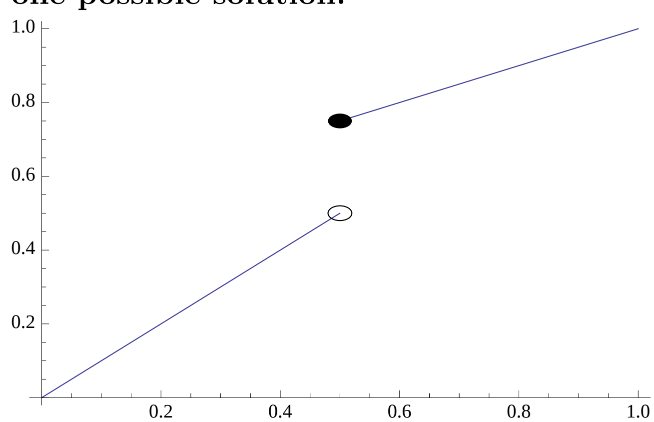
   We have $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and so $f'(5) = \frac{1}{3} \cdot 5^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{25} = \frac{1}{75}$

   We get $126^{\frac{1}{3}} \approx 5 + 1 \cdot \frac{1}{75}$.

7. Draw examples of functions with domain $(0, 1)$ of the following types. Explain clearly why your examples have the required properties.

   (a) $y = f(x)$ is increasing and not continuous

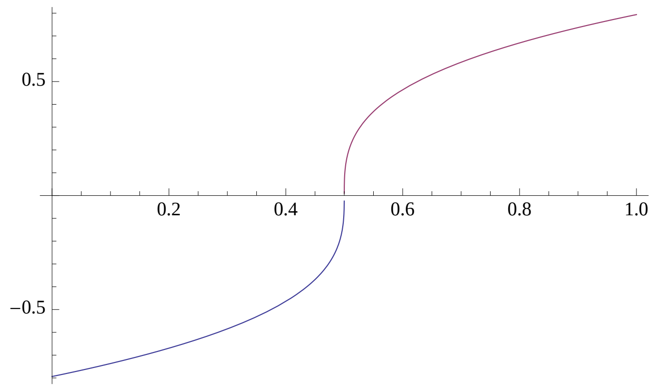
   one possible solution:



This is a graph of the function $f(x) = \begin{cases} x & \text{if } 0 \leq x < 0.5 \\ \frac{x}{2} + 0.5 & \text{if } 0.5 \leq x \leq 1 \end{cases}$
It has positive slope and is defined on the whole interval but is not continuous at $x = 0.5$.

(b) $y = f(x)$ is increasing, continuous, and not differentiable at some c in $(0, 1)$

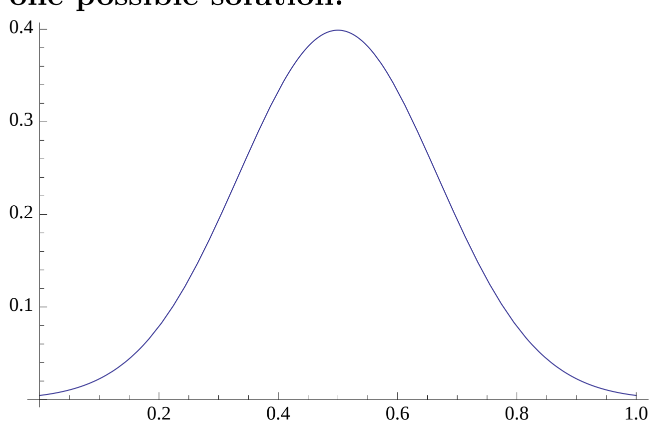
one possible solution:



This is a graph of the function $f(x) = (x - 0.5)^{\frac{1}{3}}$. It is continuous on the interval $[0, 1]$ but has a vertical slope at $x = 0.5$ so it is not differentiable there. (The slight break in the curve near the point $(0.5, 0)$ is not intentional.)

(c) $y = f(x)$ has exactly two points of inflection and exactly one relative extreme point

one possible solution:



This is the graph of a scaled Normal Distribution, centered at 0.5 and scaled so that it has all its identifying features on the domain $[0, 1]$. The graph has one local maximum at $x = 0.5$ and it has two inflection points near the tails of the distribution where the graph is concave up; near the center of the distribution the graph is concave down.