Sample Midterm 1, Math 16a, Spring 2009
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Solutions

1. **Find the points on the graph of** $y = x^5 + 10$ **where the tangent is perpendicular to the line** $y = -x/5$**.**

   Answer: $\boxed{(1, 11) \text{ and } (-1, 9)}$

   The slope of the line $y = -x/5$ is $\frac{-1}{5}$. Since we want to find a tangent line perpendicular to this line, then the slope of our tangent line must be the negative reciprocal of $\frac{-1}{5}$, which is 5.

   The slope of the tangent line is the same as the derivative, so we set the derivative equal to 5. Our graph is $f(x) = y = x^5 + 10$, so the derivative is $f'(x) = 5x^4$. By setting this equal to 5 we have $5x^4 = 5$, so $x^4 = 1$. Therefore $x = 1$ or -1.

   We plug these two values of x back into the graph, and see that if $x = 1$, then $y = 1^5 + 10 = 11$, and if $x = -1$, then $y = (-1)^5 + 10 = 9$. This gives us the two points, $(1, 11)$ and $(-1, 9)$.

2. **Find**

$$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

   Answer: $\boxed{1/6}$

   First, we try to plug 9 into the fraction. This gives us $\frac{\sqrt{9}-3}{9-9} = \frac{0}{0}$. So we must investigate further.

   We multiply both the numerator and the denominator of the fraction by $\sqrt{x} + 3$:

$$\frac{\sqrt{x} - 3}{x - 9} = \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$$

As x approaches 9, x is not equal to 9, so we may cancel the $(x-9)$ factor from the top and bottom of the fraction to see that

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3}$$

Next we simply plug in the value, $x = 9$, and see that the limit is equal to $\frac{1}{\sqrt{9}+3} = \frac{1}{6}$.

3. **Find the derivative of**

$$y = (x^5 + x^4 + x^3 + x^2 + x)^{17}$$

at $x = -1$**.**

Answer: $\boxed{51}$

We use the general power rule, that if $f(x) = g(x)^n$ then

$$f'(x) = n(g(x))^{n-1} g'(x)$$

where in our case $g(x) = (x^5 + x^4 + x^3 + x^2 + x)$ and $n = 17$. To find $g'(x)$ we use the sum rule, which states that the derivative of a sum is equal to the sum of the derivatives. Since the derivative of x^5 is $5x^4$ and the derivative of x^4 is $4x^3$, etc., we see that

$$g'(x) = (5x^4 + 4x^3 + 3x^2 + 2x + 1).$$

We plug all of this back into the general power rule formula, and see that

$$f'(x) = 17(x^5 + x^4 + x^3 + x^2 + x)^{16}(5x^4 + 4x^3 + 3x^2 + 2x + 1)$$

But we're not done yet! The problem asks for the derivative at $x = -1$. So we plug -1 into the above formula to get

$$f'(-1) = 51$$

4. **Find**

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x - 1}$$

Answer: $\boxed{\text{The limit does not exist}}$

First we try to plug 1 into the fraction. This gives us $\frac{1^2-1}{1^2-2(1)+1} = \frac{0}{0}$, so we must investigate further.

We can factor the numerator and denominator to see that

$$\frac{x^2 - 1}{x^2 - 2x + 1} = \frac{(x-1)(x+1)}{(x-1)(x-1)}.$$

So since as x approaches 1, x does not equal 1, and we may cancel an $(x-1)$ factor from both the top and the bottom. Therefore we see that

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x + 1} = \lim_{x \to 1} \frac{x+1}{x-1}$$

We now plug in $x = 1$ into the simplified fraction to get $\frac{1+1}{1-1} = \frac{2}{0}$. This is certainly undefined, and therefore the limit does not exist.

5. **Find the equation of the line which is tangent to the curve**

$$y = 3x^{1/3}$$

at the point where $x = 27$.

Answer: $\boxed{y = x/9 + 6}$

First we find the slope of the tangent line. This is the same as the derivative. The derivative of $f(x) = 3x^{1/3}$ is $f'(x) = x^{-2/3}$. So at the point where $x = 27$, the slope of the tangent line is

$$27^{-2/3} = \frac{1}{27^{2/3}} = \frac{1}{(27^{1/3})^2} = \frac{1}{3^2} = 1/9.$$

Next we find the point on the graph by plugging $x = 27$ into the graph equation to get $y = 3(27)^1/3 = 3 \cdot 3 = 9$.

Now we can construct the point-slope equation of the tangent line:

$$(y - 9) = (1/9)(x - 27)$$

and from this we get the slope-intercept form,

$$y = x/9 + 6$$

6. **Using the derivative, find an approximate value of** $15^{1/4}$**.**

   Hint: $16^{1/4} = 2$**.**

   Answer: 2 - 1/32

   We use the formula for approximating by derivative:

$$f(a + h) \approx f(a) + h \cdot f'(a)$$

   Now we find the appropriate values for f, h and a, that will help us. The function, f is certainly $f(x) = x^{1/4}$. So the derivative of this function is $f'(x) = (1/4)x^{-3/4}$. The hint indicates that the best value for a is 16. This means that our value for h will be -1. We can then rewrite the approximation formula to fit our specifications and solve:

$$\begin{aligned} (15)^{1/4} &\approx 16^{1/4} + (-1) \cdot (1/4)16^{-3/4} \\ &\approx 2 - (1/4)(\tfrac{1}{16^{3/4}}) \\ &\approx 2 - (1/4)(\tfrac{1}{(16^{1/4})^3}) \\ &\approx 2 - (1/4)(\tfrac{1}{2^3}) \\ &\approx 2 - (1/4)(1/8) \\ &\approx 2 - 1/32 \end{aligned}$$

7. **Suppose** $f(x) = |1 - x|^5$**. Find** $f'(x)$**.**

   Answer: $-5(1 - x)^4$ if $x \leq 1$, and $5(1 - x)^4$ if $x > 1$.

First thing, we rewrite the function in bracket notation:

$$f(x) = \begin{cases} (1-x)^5 & : \quad (1-x) \geq 0 \\ (-(1-x))^5 & : \quad (1-x) < 0 \end{cases}$$

If $(1-x) \geq 0$, then $x \leq 1$, and if $(1-x) < 0$, then $x > 1$. So this function can be rewritten:

$$f(x) = \begin{cases} (1-x)^5 & : \quad x \leq 1 \\ -(1-x)^5 & : \quad x > 1 \end{cases}$$

This we can differentiate for all points other than 1 directly:

$$f'(x) = \begin{cases} -5(1-x)^4 & : \quad x < 1 \\ 5(1-x)^4 & : \quad x > 1 \end{cases}$$

To see if $f'(1)$ is defined we must check both the two equations for $f(x)$ agree at $x = 1$ and that the two formulas for $f'(x)$ agree at $x = 1$.

The fomula which defines $f(x)$ for $x > 1$ is $(1-x)^5$ and the formula which defines $f(x)$ for $x < 1$ is $-(1-x)^5$. These two formulas give the same value when evaluated at $x = 1$.

The fomula which defines $f'(x)$ for $x > 1$ is $5(1-x)^4$ and the formula which defines $f'(x)$ for $x < 1$ is $-5(1-x)^4$. These two formulas give the same value when evaluated at $x = 1$.

Thus $f(x)$ is differentiable at $x = 1$ and $f'(x)$ is equal to $-5(1-x)^4$ if $x \leq 1$ and $5(1-x)^4$ if $x > 1$.

8. **Compute the derivative of $f(x) = x^3$ at $x = 1$ using the definition of the derivative as a limit.**

We recall the limit definition of the derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and plug in $f(x) = x^3$ and $a = 1$ to get

$$f'(1) = \lim_{h \to 0} \frac{(1+h)^3 - 1^3}{h}$$

If we plug in $h = 0$, we have $f'(1) = \frac{0}{0}$, so we must investigate further. First we expand $(1+h)^3$ to $(1 + 3h + 3h^2 + h^3)$. This gives us

$$f'(1) = \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} = \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{h}$$

Since h is approaching 0, then h does not equal 0, so we can cancel an h from the top and bottom of the fraction to get

$$f'(1) = \lim_{h\to 0} \frac{3 + 3h + h^2}{1} = 3 + 3h + 3h^2$$

Now when we try to substitute $h = 0$, we find that $f'(1) = 3+3(0)+(0)^2 = 3$.

So $\boxed{f'(1) = 3.}$

9. **Suppose**

$$f(x) = \frac{x^2 - 5x + 6}{x^2 - 4}$$

for $x \neq 2$ and that $f(2) = b$. Suppose $f(x)$ is differentiable at $x = 2$. Find b.

Since the function is differentiable, then it must be continuous. Therefore, $f(2) = \lim_{x\to 2} f(x)$. If we just plug $x = 2$ into our function, we get

$$f(2) = \frac{2^2 - 5(2) + 6}{2^2 - 4} = \frac{0}{0}$$

so we must investigate further. Factoring the numerator and denominator of the fraction yields

$$f(x) = \frac{(x-2)(x-3)}{(x-2)(x+2)}$$

As x approaches 2, x is not equal to 2, and so we can cancel a $(x-2)$ factor from both the top and bottom of the fraction to get

$$f(2) = \lim_{x\to 2} \frac{x-3}{x+2}$$

Now when we plug $x = 2$ into our function we get

$$f(2) = \frac{2-3}{2+2} = -1/4$$

So the value for $b = f(2)$ that will make this function continuous at $x = 2$ is $\boxed{b = -1/4.}$

10. **Find the maximum value of**

$$f(x) = (x^8/8) - (x^6/6)$$

on the interval $[-2, 1]$

First solve $f'(x) = 0$. $f'(x) = x^7 - x^5 = x^5(x^2 - 1)$. So $f'(x) = 0$ gives $x^5(x^2 - 1) = 0$. This has three solutions, $-1, 0$, and 1.

The maximum value of $f(x)$ on the interval, $[-2, 1]$, must occur at -2, 1 or where $f'(x) = 0$.

Thus the maximum value of $f(x)$ on the interval, $[-2, 1]$ must occur at -2, -1, 0 or 1.

At this point one can simply evaluate $f(x)$ at each each of these 5 vlues of x and pick the largest.

Alternatively one can examine the graph of $f(x)$ more closely.

$f'(x) = x^5(x^2 - 1)$. Solving $f'(x) > 0$ gives $x^5(x^2 - 1) > 0$ which reduces to $-1 < x < 0$ or $x > 1$.

Thus

> $f(x)$ is decreasing on $(-\infty, -1)$;
> $f(x)$ is increasing on $(-1, 0)$;
> $f(x)$ is decreasing on $(0, 1)$;
> $f(x)$ is increasing on $(1, \infty)$.

Therefore $f(x)$ has relative minimum points at $x = -1$ and $x = 1$, and a relative maximum point at $x = 0$.

Thus the maximum value of $f(x)$ on the interval, $[-2, 1]$ cannot occur at -1 or 1 (because $f(x)$ has relative minimum points there). So the maximum value of $f(x)$ on the interval, $[-2, 1]$, must occur at either $-2, 0$, or 1.

Evaluating $f(x)$ at these vaules of x:

$f(-2) = (-2)^8/8 - (-2)^6/6 = 2^6(1/2 - 1/6) = 2^6/3;$
$f(0) = (0)^8/8 - (0)^6/6 = 0;$
$f(1) = (1)^8/8 - (1)^6/6 = 1/8 - 1/6 = -1/24.$

Clearly the largest of these values is $f(-2)$.

Answer: $\boxed{2^6/3}$

11. **Find the maximum value of**

$$f(x) = (x^4 + 2)^{-3}$$

By the general power rule

$$f'(x) = (-3)(x^4 + 2)^{-4}(4x^3)$$

and $f'(x)$ is defined for all values of x.

The domain of $f(x)$ is $(-\infty, \infty)$, therefore the maximum value of $f(x)$ must occur where $f'(x) = 0$.

First solve $f'(x) = 0$ to find all possible candidates for global maximum points of $f(x)$:

$$f'(x) = 0$$

gives

$$(-3)(x^4 + 2)^{-4}(4x^3) = 0$$

which reduces to $4x^3 = 0$. This has only one solution, $x = 0$.

Let's check that $f(x)$ has a global maximum point at $x = 0$.

Note that $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$ (why?).

Thus $f(x)$ is increasing on $(-\infty, 0]$ and $f(x)$ is decreasing on $[0, \infty)$ (why?).

Therefore $f(x)$ must have a global maximum point at $x = 0$ and $f(0)$ is the maximum value of $f(x)$.

Finally $f(0) = (0^4 + 2)^{-3} = 1/8$.