# Midterm # 1, Math 16A, Fall 2012

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October 4, 2012

YOUR NAME: [ ]

YOUR GSI's NAME: [ ]

YOUR SECTION NUMBER: [ ]

This exam is multiple choice and contains 9 problems. Each of these problems is worth 11 points. Filling in your name, your GSI's name and your section number correctly is worth 1 point for a total of 100 points. There is no penalty for an incorrect answer, however you must **circle one answer for each problem** and **show your work** to get **any** credit for the correct answer. If you select an incorrect answer, partial credit is possible for these problems **if you show your work**. Important: Do all work on the exam, use the back if you need more space.

**No electronic devices, notes, or books.**

**Please do not write below here.**

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________ TOTAL

1. Find the point(s) on the graph of $y = x^7 + 3$ where the tangent line is perpendicular to the line $y = -\frac{1}{7}\,x + 3$.

   (a) $(1, 4)$ and $(-1, 2)$
   (b) $(1, 4)$
   (c) $(-1, 2)$
   (d) $y = 7x + 1$
   (e) $y = 7x - 1$
   (f) $y = 7x^6$

   SOLUTION: The derivative of the curve $y = x^7 + 3$ is $y' = 7x^6$. The line perpendicular to the line $y = -\frac{1}{7}\,x + 3$ has slope 7. Therefore, we need to find points where $y' = 7x^6 = 7$. These are satisfying $x^6 = 1$ or $x = \pm 1$.

   For $x = 1$, $y = 1^7 + 3 = 4$. For $x = -1$, $y = (-1)^7 + 3 = 2$. Thus, the answer is (a).

2. Suppose $f(x) = |x+1| + |x-3|^3$. Find the domain of $f'(x)$.

   (a) $x \neq -1$

   (b) $x \neq 0$

   (c) $-\infty < x < \infty$

   (d) $0 \leq x < \infty$

   (e) $x \neq 3$ and $x \neq -1$

   (f) $x \neq 3$

SOLUTION:
f(x) is piecewise continuous with possible changes of formula at $x = -1$ and $x = 3$. In fact,

$$|x+1| = \begin{cases} (-1-x) & \text{if} \quad x < -1 \\ (x+1) & \text{if} \quad -1 \leq x \end{cases}$$

and

$$|x-3| = \begin{cases} (3-x) & \text{if} \quad x < 3 \\ (x-3) & \text{if} \quad 3 \leq x \end{cases}$$

so

$$f(x) = \begin{cases} (-1-x) + (3-x)^3 & \text{if} \quad x < -1 \\ (x+1) + (3-x)^3 & \text{if} \quad -1 \leq x < 3 \\ (x+1) + (x-3)^3 & \text{if} \quad 3 \leq x \end{cases}$$

Then

$$f'(x) = \begin{cases} (-1) + 3(3-x)^2(-1) & \text{if} \quad x < -1 \\ \textbf{TBD} & \text{if} \quad x = -1 \\ (1) + 3(3-x)^2(-1) & \text{if} \quad -1 < x < 3 \\ \textbf{TBD} & \text{if} \quad x = 3 \\ (1) + 3(x-3)^2(-1) & \text{if} \quad 3 < x. \end{cases}$$

Since

$$\lim_{x \to -1} (-1) + 3(3-x)^2(-1) = -49 \neq -47 = \lim_{x \to -1} (1) + 3(3-x)^2(-1),$$

we see that $f'(-1)$ is undefined. Since

$$\lim_{x \to 3} (1) + 3(3-x)^2(-1) = 1 = \lim_{x \to 3} (1) + 3(x-3)^2(-1),$$

we see that $f'(3)$ is defined and equals 1. Hence the domain of $f'(x)$ is

$$\{x \neq -1\} = (-\infty, -1) \cup (-1, \infty).$$

3. Find the derivative of

$$y = \left(2 + 3x + x^3 + \frac{1}{x^4}\right)^9$$

at $x = -1$.

(a) $9(2 + 3x + x^3 + \frac{1}{x^4})^8(3 + 3x^2 - 4x^{-5})$

(b) $9(2 + 3x + x^3 + \frac{1}{x^4})^8$

(c) 0

(d) 90

(e) $-90$

(f) The derivative does not exist at $x = -1$.

SOLUTION:

Using the general power rule, we find that

$$y' = 9(2 + 3x + x^3 + \frac{1}{x^4})^8(3 + 3x^2 - 4x^{-5})$$

And at $x = -1$,

$$\begin{aligned} y' &= 9(2 + 3(-1) + (-1)^3 + \frac{1}{(-1)^4})^8(3 + 3(-1)^2 - 4(-1)^{-5}) \\ &= 9(2 - 3 - 1 + 1)^8(3 + 3 + 4) \\ &= 9(-1)^8(10) \\ &= 90 \end{aligned}$$

4. Using the derivative, find an approximate value of $(15)^{1/2}$.

   (a) 3
   (b) $3\frac{7}{8}$
   (c) $4\frac{1}{8}$
   (d) 4
   (e) $4\frac{3}{16}$
   (f) $3\frac{1}{16}$

   SOLUTION:

   Use formula $f(a+h) \approx f(a) + h \cdot f'(a)$, where $f(x) = (x)^{\frac{1}{2}}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ , $a = 16$ and $h = -1$. Then $(15)^{1/2} = f(16 + (-1)) \approx f(16) + (-1) \cdot f'(16) = (16)^{\frac{1}{2}} - \frac{1}{2} \cdot 16^{-\frac{1}{2}} = 4 - \frac{1}{2} \cdot \frac{1}{4} = 4 - \frac{1}{8} = 3\frac{7}{8}$.

5. Find the equation of the line which is tangent to the curve

$$y = x^3$$

at the point $x = 1$.

(a) $3$
(b) $y = 3x + 1$
(c) $(1, 1)$
(d) $y = 3x - 2$
(e) $y = x$
(f) $x = 1$

SOLUTION:

The slope of the tangent line is $y'|_{x=1} = 3x^2|_{x=1} = 3$. Since $y|_{x=1} = 1$, the tangent line pass through $(1, 1)$. So the equation of the tangent line is $y - 1 = 3(x - 1)$. After simplification, this becomes $y = 3x - 2$.

6. Suppose $f(x) = \begin{cases} \frac{x^2+2x-8}{x^2-4} & \text{if } x \neq 2 \\ b & \text{if } x = 2 \end{cases}$ and $f$ is differentiable.

Find $b$. Explain your reasoning.

(a) 1
(b) 2
(c) 6/4
(d) 2/3
(e) $-1/3$
(f) $-1$

SOLUTION:

Since $f$ is differentiable it is continuous, so we must have $b = f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2+2x-8}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x+4)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x+4}{x+2} = \frac{6}{4}$

7. Find the absolute minimum value of

$$f(x) = (-x^2 + x - 1)^{-1}$$

(a) $-4/3$
(b) $-3/4$
(c) $-7/4$
(d) $0$
(e) $1/2$
(f) $-4/7$

SOLUTION:

First we compute the derivative:

$$f'(x) = -(-x^2 + x - 1)^{-2}(-2x + 1) = \frac{2x - 1}{(-x^2 + x - 1)^2}$$

The critical values are when the derivative is equal to zero, or undefined.
The derivative is zero, if the numerator is zero, so if $2x - 1 = 0$. This leads to the critical value $x = \frac{1}{2}$.
The derivative is undefined, if the denominator is zero, so if $(-x^2 + x - 1)^2 = 0$, which is impossible, since the formula has a square root of a negative number $x = \frac{-1 \pm \sqrt{1-4}}{-2}$.

So the only critical value is $x = \frac{1}{2}$. We need to verify that it is a local minimum. We do this using the first derivative test. We compute two test values, one on each side of the critical value.

$$f'(0) = \frac{2 \cdot 0 - 1}{(-0^2 + 0 - 1)^2} = \frac{-1}{(-1)^2} = -1$$

and

$$f'(1) = \frac{2 \cdot 1 - 1}{(-1^2 + 1 - 1)^2} = \frac{1}{(-1)^2} = 1$$

The function $f$ switches from being decreasing to being increasing at $x = 0$. This shows that $x = 0$ is a local minimum. Since the function has no other critical values and no discontinuities, the absolute minimum must occur at the only local minimum. So we compute $f(0) = (-(\frac{1}{2})^2 + \frac{1}{2} - 1)^{-1} = (\frac{-3}{4})^{-1} = \frac{4}{-3}$. Hence answer A.

8. Suppose that

$$f(x) = a + 3x^4 - 4x^3.$$

For which values of $a$ is there a value for $x$ such that $f(x) \leq 0$. Hint: $f$ has an absolute minimum point.

(a) $a \geq 1$
(b) $a \leq 1$
(c) $a \geq 0$
(d) $a \leq 0$
(e) $a \geq -1$
(f) $a \leq -1$

SOLUTION:

We'll use the first derivative test to find the critical points. We have

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1),$$

so the critical points are at $x = 0$ and $x = 1$.

Then, using the second derivative test, since

$$f''(x) = 36x^2 - 24x,$$

we have $f''(0) = 0$ and $f''(1) = 12$, so there is a relative minimum at $x = 1$. We know that this is in fact an absolute minimum, either because of the hint, or by further analysis of the function. (Both as $x \to \infty$ and as $x \to -\infty$, we have $f(x) \to \infty$, and the only critical points are an inflection point at $x = 0$ and a relative minimum at $x = 1$, so overall, the relative minimum is the absolute minimum.)

The value of the function at the absolute minimum is $f(1) = a-1$. The question is asking for values of $a$ which will make *some part* of the function be less than or equal to zero. Since $f(1)$ is the minimum value, this is the same as asking for values of $a$ which will make $f(a)$ less than or equal to zero. Thus, we want $a$ for which $f(a) = a - 1 \leq 0$, which is equivalent to $a \leq 1$. Thus, the correct answer is (b).

**Remark**: For this question, a large portion of the marks were given for the quality and completeness of the explanations. So for instance, solutions that contained nothing but equations and numbers typically scored no more than 6 out of 11.

9. Compute

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h}$$

(a) $-1/16$

(b) $-1/32$

(c) $0$

(d) $1/16$

(e) $1/32$

(f) The limit does not exist.

SOLUTION 1:

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}}{h} = \lim_{h\to 0} \frac{2-\sqrt{4+h}}{h2\sqrt{4+h}} = \lim_{h\to 0} \frac{2-\sqrt{4+h}}{2h\sqrt{4+h}} \frac{(2+\sqrt{4+h})}{(2+\sqrt{4+h})}$$

$$= \lim_{h\to 0} \frac{4-(4+h)}{2h\sqrt{4+h}(2+\sqrt{4+h})} = \lim_{h\to 0} \frac{-h}{2h\sqrt{4+h}(2+\sqrt{4+h})} = \lim_{h\to 0} \frac{-1}{2\sqrt{4+h}(2+\sqrt{4+h})} = -\frac{1}{16}.$$

SOLUTION 2:

You can recognize this limit as a derivative. In fact,

$$L = \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = f'(4)$$

where $f(x) = \frac{1}{\sqrt{x}}$. Using the power rule, we conclude that $f'(x) = -\frac{1}{2}x^{-3/2}$, therefore $L = f'(4) = -\frac{1}{16}$.