Solution
by Benjamin Johnson

1. Find the point on the graph of $y = x^2$ where the curve has slope -6.

$$y' = 2x$$

$$2x = 6$$

$$\Rightarrow \quad x = 3$$

$$\Rightarrow \quad y = 3^2 = 9$$

The point is $\boxed{(3,9)}$

2. Evaluate the following limit.

$$\lim_{x \to 0} \frac{\ln(2+x) - \ln(2)}{x}$$

$$\lim_{h \to 0} \frac{\ln(2+h) - \ln(2)}{h}$$

is the definition of $f'(2)$
where $f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f'(2) = \boxed{\frac{1}{2}}$

3. Differentiate the following functions.

(a) $f(x) = x^x$

(b) $g(x) = \left(2\ln(x+2)+1\right)^{200}$

(a) $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln x]$

$\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x}$

$\frac{dy}{dx} = y(\ln x + 1)$

$= \boxed{x^x(\ln x + 1)}$

(b) $g'(x) = 200(2\ln(x+2)+1)^{199}$

$\cdot\, 2 \cdot \frac{1}{x+2}$

$= \boxed{\frac{400(2\ln(x+2)+1)^{199}}{x+2}}$

4. Find the equation of the line tangent to the curve $y^2 = x^3 + x + 6$ at the point $(2, 4)$.

$$y^2 = x^3 + x + 6$$

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[x^3 + x + 6\right]$$

$$2y\frac{dy}{dx} = 3x^2 + 1$$

$$\frac{dy}{dx} = \frac{3x^2+1}{2y}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=4}} = \frac{3\cdot(2)^2+1}{2\cdot 4}$$

$$= \frac{13}{8}$$

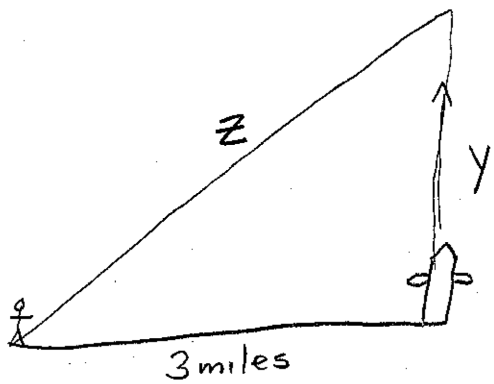
This is the slope of the tangent line.

The equation is: $y - 4 = \frac{13}{8}(x-2)$

$$y = \frac{13}{8}x - \frac{13}{4} + \frac{16}{4}$$

$$\boxed{y = \frac{13}{8}x + \frac{3}{4}}$$

5. You are viewing the launch of the space shuttle from a safe distance of 3 miles from the launch pad. Find the vertical speed of the shuttle at the instant when the distance between you and the shuttle is 5 miles and that distance is increasing at 5,000 miles/hour.



Find $\frac{dy}{dt}$ when $z = 5_{\text{miles}}$ and $\frac{dz}{dt} = 5000 \text{ miles/hr}$

$$3^2 + y^2 = z^2 \implies \text{when } z = 5, \quad 9 + y^2 = 25, \quad y^2 = 16, \quad y = 4$$

$$\frac{d}{dt}\left[3^2 + y^2\right] = \frac{d}{dt}\left[z^2\right]$$

$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\frac{dy}{dt} = \frac{z}{y}\frac{dz}{dt}$$

$$\left.\frac{dy}{dt}\right|_{\substack{\frac{dz}{dt} = 5000 \\ z = 5 \\ y = 4}} = \frac{5}{4} \cdot 5000 = \boxed{6250 \text{ miles/hour}}$$

6. Use a linear approximation to give an approximate value for $1001^{\frac{1}{3}}$.

Let $f(x) = x^{1/3}$, $a = 1000$, $h = 1$

$$f(a+h) \approx f(a) + hf'(a)$$

$$(1001)^{1/3} \approx (1000)^{1/3} + 1 \cdot \tfrac{1}{3}(1000)^{-2/3}$$

$$= 10 + \tfrac{1}{3} \cdot \left[(1000)^{1/3}\right]^{-2}$$

$$= 10 + \tfrac{1}{3} \cdot \frac{1}{100}$$

$$= 10 + \frac{1}{300}$$

$$= \boxed{\frac{3001}{300}}$$

7. Evaluate the following indefinite integrals.

(a) $\int xe^{x^2}\,dx$

(b) $\int \frac{1}{x}\,dx$ (Note, $\frac{1}{x}$ has domain the set x with $x \neq 0$.)

(a) $\left(e^{x^2}\right)' = e^{x^2} \cdot 2x = 2xe^{x^2}$

This is almost it.

Try $\left[\frac{1}{2}e^{x^2}\right]' = \frac{1}{2} \cdot e^{x^2} \cdot 2x = xe^{x^2}$ ✓

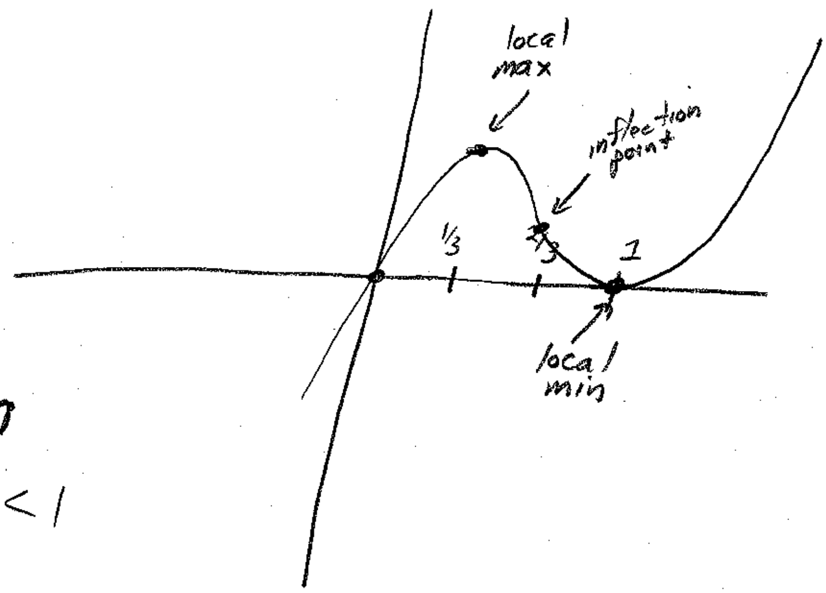
so $\int xe^{x^2}dx = \boxed{\frac{1}{2}e^{x^2} + C}$

(b) $\boxed{\ln|x| + C}$

8. Sketch the graph of $y = x^3 - 2x^2 + x$. Clearly indicate all x-intercepts, relative and absolute extreme points, and points of inflection.

Domain: $(-\infty, \infty)$

$$f(x) = x^3 - 2x^2 + x$$
$$= x(x^2 - 2x + 1)$$
$$= x(x-1)^2$$



x-intercepts: 0, 1

$$f(x) \text{ is } \begin{cases} < 0 & \text{if } x < 0 \\ > 0 & \text{if } 0 < x < 1 \\ > 0 & \text{if } x > 1 \end{cases}$$

$$f'(x) = 3x^2 - 4x + 1$$
$$= (3x-1)(x-1)$$

$f'(x) = 0$ when $x = 1/3, 1$

$$f'(x) \text{ is } \begin{cases} > 0 & \text{if } x < 1/3 \\ < 0 & \text{if } 1/3 < x < 1 \\ > 0 & \text{if } x > 1 \end{cases}$$

$$f''(x) = 6x - 4$$
$$= 6(x - 2/3)$$

$f''(x) = 0$ when $x = 2/3$

$$f''(x) \text{ is } \begin{cases} < 0 & \text{if } x < 2/3 \\ > 0 & \text{if } x > 2/3 \end{cases}$$

9. Sketch the graph of $y = xe^{-x^2}$. Clearly indicate all x-intercepts, relative and absolute extreme points, and points of inflection.

Domain: $(-\infty, \infty)$

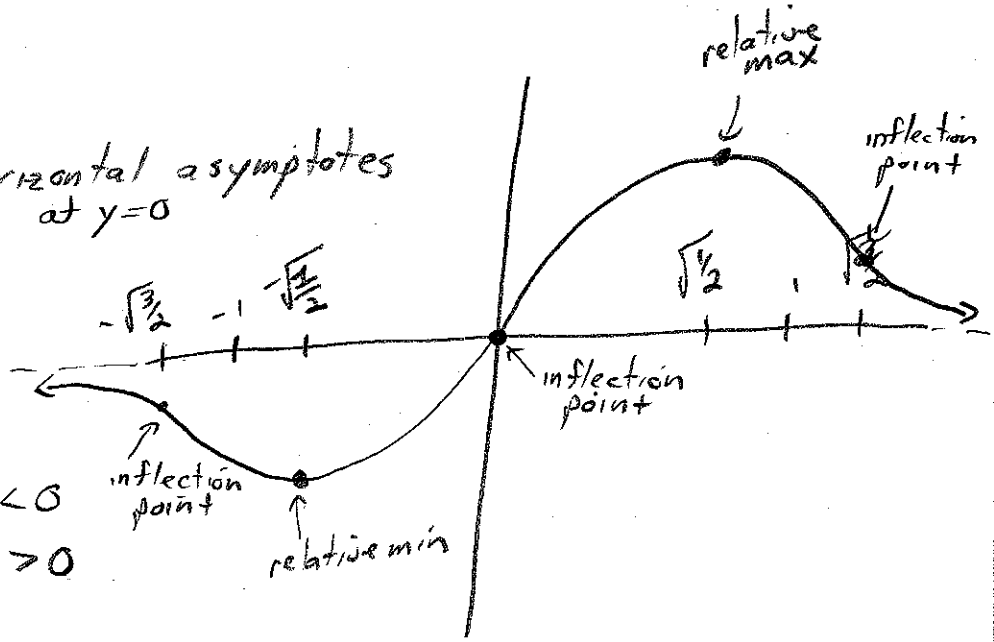
$\lim_{x \to \infty} xe^{-x^2} = 0$ ← horizontal asymptotes at $y=0$

$\lim_{x \to -\infty} xe^{-x^2} = 0$

$f(x) = xe^{-x^2}$

$f(x) = 0$ when $x = 0$

$f(x)$ is $\begin{cases} <0 & \text{when } x<0 \\ >0 & \text{when } x>0 \end{cases}$



$$f'(x) = e^{-x^2} + x \cdot e^{-x^2} \cdot -2x$$
$$= e^{-x^2} - 2x^2 e^{-x^2}$$
$$= (1-2x^2)e^{-x^2}$$

$f'(x) = 0$ when $x = \pm \frac{1}{\sqrt{2}}$

$f'(x)$ is $\begin{cases} <0 & \text{if } x < -\frac{1}{\sqrt{2}} \\ >0 & \text{if } -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \\ <0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$

$$f''(x) = -4xe^{-x^2} + (1-2x^2) \cdot -2xe^{-x^2}$$

$$= -4xe^{-x^2} + (-2x + 4x^3)e^{-x^2}$$
$$= (-6x + 4x^3)e^{-x^2}$$
$$= 6x\left(\tfrac{2}{3}x^2 - 1\right)e^{-x^2}$$

$f''(x) = 0$ when $x = 0, -\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}$

$f''(x)$ is $\begin{cases} <0 & \text{if } x < -\sqrt{3/2} \\ >0 & \text{if } -\sqrt{3/2} < x < 0 \\ <0 & \text{if } 0 < x < \sqrt{3/2} \\ >0 & \text{if } x > \sqrt{3/2} \end{cases}$

10. Calculate the volume of the solid of revolution obtained by rotating the region under graph $y = \frac{1}{\sqrt{x}}$ between 1 and 2 about the x-axis.

$$\int_1^2 \pi \left(\frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int_1^2 \frac{\pi}{x}\, dx$$

$$= \pi \ln x \Big]_1^2$$

$$= \pi [\ln 2 - \ln 1]$$

$$= \boxed{\pi \ln 2}$$

11. Determine all functions $y = f(x)$ such that $y' = -.5y$ and $f(0) = 1$.

$$\boxed{f(x) = \underline{1}\,e^{-0.5x}}$$

12. An investor initially invests \$10,000 in a speculative venture. Suppose that the investment earns 20% interest compounded continuously for the first 5 years and then 6% interest compounded continuously for 5 years thereafter. How much is the investment worth after 10 years?

Let $f(t)$ = amount after t years

$$f(5) = 10000\, e^{0.2 \cdot 5}$$
$$= 10000\, e$$

$$f(10) = 10000\, e \cdot e^{0.06 \cdot 5}$$
$$= 10000\, e \cdot e^{0.3}$$
$$= \boxed{10000\, e^{1.3} \text{ dollars}}$$