MATH 16A, SUMMER 2008, FINAL EXAM SOLUTION

BENJAMIN JOHNSON

(1) (6 points) [Implicit Differentiation] Write the equation of the line that is tangent to the curve $x^2 + y^2 = 4$ at the point $(1, -\sqrt{3})$.

The slope of this line is $\frac{dy}{dx}|_{x=1;y=-\sqrt{3}}$. We have

$$\begin{aligned}
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(4) \\
2x + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{x}{y} \\
\frac{dy}{dx}|_{x=1;y=-\sqrt{3}} &= -\frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}
\end{aligned}$$

The equation of the line is $y - (-\sqrt{3}) = \frac{\sqrt{3}}{3}(x - 1)$, which simplifies to $y = \frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{3}$

(2) (6 points) [Related Rates] A 10-foot-long ladder is leaning against the side of a brick wall. If the top of of the ladder is sliding down the wall vertically at a constant rate of 2 feet per second, how fast is the ladder sliding along the ground away from the wall when the top of the ladder is 6 feet from the ground?

Let y be the height of the ladder and let x be the distance of the ladder from the wall. We are given that $\frac{dy}{dt} = -2$ and are asked to find $\frac{dx}{dt}|_{y=6}$. Since the ladder, the floor, and the wall make a right triangle, we have $x^2 + y^2 = 10^2$. Taking the derivative with respect to t of both sides of this equation, we obtain $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, and hence $\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$. When $y = 6$, we have $x = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$; so $\frac{dx}{dt}|_{y=6;x=8;\frac{dy}{dx}=-2} = -\frac{6}{8} \cdot -2 = \frac{3}{2}$ feet per second.

(3) (18 points) [Differentiation] Compute the derivative of the following functions. Simplify your answer.

(a) $y = x^2 + 2x + 75$

$y' = 2x + 2$

(b) $f(t) = (2t^2 + 4)^6$

$f'(t) = 6(2t^2 + 4)^5 \cdot 4t = 24t(2t^2 + 4)^5$

(c) $R = \frac{\ln d}{d^2}$

$R' = \frac{\frac{1}{d}(d^2) - \ln d(2d)}{(d^2)^2} = \frac{d(1-2\ln d)}{d^4} = \frac{1-2\ln d}{d^3}$

(d) $g(y) = (2y - 1)e^y$

$g'(y) = 2 \cdot e^y + (2y - 1) \cdot e^y = (2 + 2y - 1)e^y = (2y + 1)e^y$

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(e) $r(w) = \frac{(3w-2)(4w+4)(w^2)(5w-9)}{2w+1}$

Use $r'(w) = r(w) \cdot \frac{d}{dw} \ln r(w)$.
We have $\ln r(w) = \ln(3w-2) + \ln(4w+4) + 2\ln w + \ln(5w-9) - \ln(2w+1)$,
and $\frac{d}{dw} \ln r(w) = \frac{3}{3w-2} + \frac{4}{4w+4} + \frac{2}{w} + \frac{5}{5w-9} - \frac{2}{2w+1}$.
So $r'(w) = \frac{(3w-2)(4w+4)(w^2)(5w-9)}{2w+1} \cdot \left(\frac{3}{3w-2} + \frac{4}{4w+4} + \frac{2}{w} + \frac{5}{5w-9} - \frac{2}{2w+1}\right)$

(f) $z(x) = x^x$

Use $z'(x) = z(x) \cdot \frac{d}{dx} \ln z(x)$.
We have $\ln z(x) = x \ln x$,
and $\frac{d}{dx} \ln z(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$.
So $z'(x) = x^x(\ln x + 1)$

(4) (6 points) [Exponents and Logarithms] Solve for x.

(a) $e^{2x} + 2(e^x) = 3$

$(e^x)^2 + 2e^x - 3 = 0$
$(e^x + 3)(e^x - 1) = 0$
So $e^x = -3$ or $e^x = 1$. The equation $e^x = -3$ has no solution. The solution to $e^x = 1$ is $x = 0$.

(b) $\ln(x+1) - \ln(x-2) = 1$.

$\ln \frac{x+1}{x-2} = 1$
$\frac{x+1}{x-2} = e$
$x + 1 = e(x-2)$
$x - ex = -1 - 2e$
$x = \frac{-1-2e}{1-e} = \frac{2e+1}{e-1}$

(5) (12 points) [Exponential Growth] A certain community of insects is growing at a rate that is 4 times the current insect population. Suppose that at time $t = 0$ (where t is in weeks) there are 1000 insects.

(a) Find a formula for $P(t)$, the insect population at time t.

$P(t) = 1000e^{4t}$

(b) What is the population at time $t = 5$?

$P(5) = 1000e^{20}$ insects.

(c) How long will it take the insect population to triple?

$3000 = 1000e^{4t}$
$3 = e^{4t}$
$\ln 3 = 4t$
$t = \frac{\ln 3}{4}$ weeks

(d) How many insects will there be when the population is increasing at a rate of 18000 insects per week?

Use $P'(t) = 4P(t)$. When $P'(t) = 18000$, $P(t)$ will be $\frac{18000}{4} = 4500$ insects.

(6) (12 points) [Compound Interest]

Suppose that you invest 400 in a savings account that earns 5% interest compounded continuously.

(a) Write a formula that gives the amount of money you can expect to have t years after making this deposit.

$P(t) = 400e^{.05t}$

(b) How much money will you have after 2 years?
$P(2) = 400e^{.1}$ dollars

(c) How long do you have to wait until you have $400e$ dollars?

Solve $P(t) = 400e$ for t.
$400e^{.05t} = 400e$
$t = 20$ years

(d) At what rate will your money be increasing when you have \$1000 in your account?

Use $P'(t) = .05P(t)$. When $P(t) = 1000$, $P'(t)$ will be 50 dollars per year

(7) (6 points) [Learning Curve] The percentage of calculus material from this course that you know is related to the amount of time that you study. Suppose that this percentage could be approximated by the learning curve function $f(t) = 100(1 - e^{-.04t})$ where t is in hours of study time.

(a) If you wanted to learn 80% of the material, how long should you have studied?

Solve $f(t) = 80$ for t.
$100(1 - e^{-.04t}) = 80$
$1 - e^{-.04t} = \frac{4}{5}$
$\frac{1}{5} = e^{-.04t}$
$\ln \frac{1}{5} = -.04t$
$t = \frac{\ln \frac{1}{5}}{-.04} = 25 \ln 5$ hours

(b) If you decided to keep studying until your rate of learning falls to 1% per hour, what percentage of the material would you have learned?

Use the differential equation for the learning curve, $f'(t) = .04(100 - f(t))$. Solving for $f(t)$, we get $f(t) = 100 - 25f'(t)$. So when $f'(t) = 1$, we have $f(t) = 100 - 25 = 75\%$.

You could also get the same answer with more work by computing $f'(t)$, solving the equation $f'(t) = 1$ for t, and then plugging this answer for t into $f(t)$.

(8) (6 points) [Antidifferentiation]

(a) Compute $\int \left(e^{3x} + \frac{3}{x^2} - \sqrt{x}\right) dx$

$\frac{1}{3}e^{3x} + x^3 - \frac{2}{3}x^{\frac{3}{2}} + C$

(b) Suppose $f'(x) = e^{2x+1}$ and $f(2) = 5$. Find a formula for $f(x)$.

$f(x) = \frac{1}{2}e^{2x+1} + C$. Since $5 = \frac{1}{2}e^{2\cdot 2+1} + C$, we have $C = 5 - \frac{1}{2}e^5$. So $f(x) = \frac{1}{2}e^{2x+1} + 5 - \frac{1}{2}e^5$.

(9) (5 points) [Fundamental Theorem of Calculus] Let $F(x) = \int_6^x (t^2 + 7)dt$, and let $h(x) = 3x^2$. Compute $(F \circ h)'(1)$.

By a version of the fundamental theorem of calculus, we have $F'(x) = x^2 + 7$. Also we have $h'(x) = 6x$. By the chain rule, $(F \circ h)'(x) = F'(h(x)) \cdot h'(x) = ((3x^2)^2 + 7) \cdot 6x = 6x(9x^4 + 7)$. So$(F \circ h)'(1) = 6 \cdot 1(9 \cdot 1^4 + 7) = 6(16) = 96$.

(10) (6 points) [The Definite Integral]
Compute the following definite integrals.

(a) $\int_0^3 (e^{2x}+1)dx$
$= \left[\frac{1}{2}e^{2x}+x\right]_0^3 = \frac{1}{2}e^{2\cdot 3}+3-\left(\frac{1}{2}e^0+0\right) = \frac{1}{2}e^6+\frac{5}{2}$

(b) $\int_1^4 (3x^2+2x+1)dx$
$= [x^3+x^2+x]_1^4 = 4^3+4^2+4-(1^3+1^2+1) = 64+16+4-3 = 81$

(11) (6 points) [Areas between curves] Find the area between the curves $y = x^2$ and $y = -2-x^2$ from $x = -1$ to $x = 1$.

$\int_{-1}^1 (x^2-(-2-x^2))dx = \int_{-1}^1 (2x^2+2)dx = 2\int_0^1 (2x^2+2)dx = 2[\frac{2}{3}x^3+2x]_0^1 = 2[(\frac{2}{3}+2)-0] = \frac{16}{3}$

(12) (5 points) [Average Value of a Function] What is the average value of the function $H(x) = e^x+3x^2+1$ from $x = 0$ to $x = 2$?

$\frac{1}{2-0}\int_0^2 (e^x+3x^2+1)dx = [e^x+x^3+x]_0^2 = e^2+2^3+2-(e^0+0+0) = e^2+8+2-1 = e^2+9$

(13) (6 points) [Volume] Find the volume of the solid that results form rotating the curve $y = e^x$ about the x-axis from $x = 0$ to $x = 1$.

$\int_0^1 \pi(e^x)^2dx = \pi\int_0^1 e^{2x}dx = \pi\left[\frac{1}{2}e^{2x}\right]_0^1 = \pi\left(\frac{1}{2}e^2-\frac{1}{2}e^0\right) = \frac{\pi}{2}(e^2-1)$