Solutions: Final Exam, Math 16a, Fall 2010 (T. Slaman)

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Problem 1 Evaluate the following limits.

(a) $\lim_{h\to 0} \frac{\ln(2+h)-\ln(2)}{h}$

Solution: By the limit definition of the derivative, we see that this limit is the derivative of $\ln(x)$ at $x = 2$. $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$, so

$$\lim_{h\to 0} \frac{\ln(2+h)-\ln(2)}{h} = \frac{1}{2}$$

(b) $\lim_{x\to 1} \frac{x^3-x}{x-1}$

Solution:

$$\begin{aligned}\lim_{x\to 1} \frac{x^3-x}{x-1} &= \lim_{x\to 1} \frac{x(x^2-1)}{x-1} \\ &= \lim_{x\to 1} \frac{x(x+1)(x-1)}{x-1} \\ &= \lim_{x\to 1} x(x+1) \\ &= 1(1+1) \\ &= 2\end{aligned}$$

Problem 2 Differentiate the following functions.

(a) $f(x) = x^x$

Solution: We note that $x^x = e^{\ln(x^x)} = e^{x\ln(x)}$, so

$$\begin{aligned}\frac{d}{dx}[x^x] &= \frac{d}{dx}\left[e^{x\ln(x)}\right] \\ &= e^{x\ln(x)}(1\cdot\ln(x) + x\cdot\frac{1}{x}) \\ &= e^{\ln(x^x)}(\ln(x)+1) \\ &= x^x(\ln(x)+1)\end{aligned}$$

Alternatively, recall that $f'(x) = f(x) \cdot \frac{d}{dx}[\ln(f(x))]$ (i.e., use logarithmic differentiation).

(b) $g(x) = (2\ln(x+2)+1)^{200}$

Solution:

$$\begin{aligned} g'(x) &= 200(2\ln(x+2)+1)^{199} \cdot \frac{d}{dx}[2\ln(x+2)+1] \\ &= 200(2\ln(x+2)+1)^{199}\left(2 \cdot \frac{1}{x+2}\right) \\ &= \frac{400}{x+2}(2\ln(x+2)+1)^{199} \end{aligned}$$

Problem 3 Find the equation of the line tangent to the curve $y^2 = x^3 + x + 6$ at the point $(2,4)$.

Solution: We first find $\frac{dy}{dx}$:

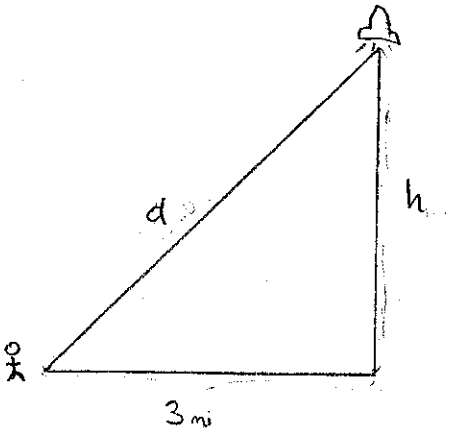
$$\begin{aligned} \frac{d}{dx}[y^2] &= \frac{d}{dx}[x^3+x+6] \\ 2y\frac{dy}{dx} &= 3x^2+1 \\ \frac{dy}{dx} &= \frac{3x^2+1}{2y} \end{aligned}$$

At the point $(2,4)$, the slope of the curve is: $\frac{3 \cdot 2^2+1}{2 \cdot 4} = \frac{13}{8}$. The equation of the tangent line is:

$$\begin{aligned} y-4 &= \frac{13}{8}(x-2) \\ y &= \frac{13}{8}x - \frac{13}{4} + 4 \\ &= \frac{13}{8}x + \frac{3}{4} \end{aligned}$$

Problem 4 You are viewing the launch of the space shuttle at a safe distance of 3 miles from the launch pad. Find the vertical speed of the shuttle at the instant when the distance between you and the shuttle is 5 miles and that distance is increasing at 5,000 miles/hour.

Solution: We can represent this situation by drawing a right triangle: the horizontal leg is the distance between the viewing point and the launch pad (3 miles); the vertical leg is the height of the space shuttle, and the hypotenuse is the distance between the viewing point and the space shuttle (5 miles at the moment described in the statement of the problem). By the Pythagorean theorem, the height of the shuttle at this moment is 4 miles ($3^2+4^2=5^2$). Both the distance between the viewing point and the shuttle ('d') and the height of the shuttle ('h') are changing, while the distance between the viewing point and the launch pad is fixed at 3 miles. We can figure out how these rates of change relate to each other using the relationship between d and h and implicit differentiation (with respect to time, t):



$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[3^2 + h^2] &= \frac{\mathrm{d}}{\mathrm{d}t}[d^2]\\ 2h \cdot \frac{\mathrm{d}h}{\mathrm{d}t} &= 2d\frac{\mathrm{d}d}{\mathrm{d}t}\\ \frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{2d\frac{\mathrm{d}d}{\mathrm{d}t}}{2h}\\ &= \frac{d}{h}\cdot\frac{\mathrm{d}d}{\mathrm{d}t}\end{aligned}$$

We know that at the moment described, $d = 5$, $h = 4$, and $\frac{\mathrm{d}d}{\mathrm{d}t} = 5000$. Thus, the vertical speed of the shuttle is:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{5}{4}\cdot 5000 = 6250 \text{ miles/hour}$$

Problem 5 Use a linear approximation to give an approximate value for $1001^{\frac{1}{3}}$.

Solution: We consider the function $f(x) = x^{\frac{1}{3}}$. Note that $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. We know that for h small, $f(a+h) \approx f(a) + hf'(a)$. Let $a = 1000$ and $h = 1$. Then we have:

$$\begin{aligned}f(1000+1) &\approx 10 + 1\cdot\frac{1}{3}\cdot 1000^{-\frac{2}{3}}\\ 1001^{\frac{1}{3}} &\approx 10 + \frac{1}{3}\cdot\frac{1}{\sqrt[3]{1000}^2}\\ &\approx 10 + \frac{1}{3}\cdot\frac{1}{10^2}\\ &\approx 10 + \frac{1}{300}\end{aligned}$$

Problem 6 Evaluate the following indefinite integrals.

(a) $\int xe^{x^2}\,\mathrm{d}x$

Solution: $\int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}e^{x^2} + C$

(b) $\int \frac{1}{x}\,\mathrm{d}x$ (Note, $\frac{1}{x}$ has domain all real numbers except for 0.)

Solution: $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$. (We need the absolute value to make the domain of the anti-derivative match the domain of the original function.)

Problem 7 Sketch the graph of $y = x^3 - 2x^2 + x$. Clearly indicate all x-intercepts, relative and absolute extreme points, and points of inflection.

Solution:

1. x-intercepts:

$$\begin{aligned} x^3 - 2x^2 + x &= 0 \\ x(x^2 - 2x + 1) &= 0 \\ x(x-1)^2 &= 0 \\ x = 0 \text{ or } x &= 1 \end{aligned}$$

2. Extrema:

   (a) We find the derivatives of y:

$$\begin{aligned} y' &= 3x^2 - 4x + 1 \\ y'' &= 6x - 4 \end{aligned}$$

   (b) We find the critical points:

$$\begin{aligned} 0 &= 3x^2 - 4x + 1 \\ &= (3x-1)(x-1) \\ x &= 1 \text{ or} \\ 3x - 1 &= 0 \\ x &= \frac{1}{3} \end{aligned}$$

   (c) We use the second derivative test to evaluate these critical points:

$$\begin{aligned} y''(1) &= 6(1) - 4 = 2 > 0 \\ y''\left(\frac{1}{3}\right) &= 6\left(\frac{1}{3}\right) - 4 = 2 - 4 = -2 < 0 \end{aligned}$$

   (d) We find the corresponding y-values

$$\begin{aligned} y(1) &= 0 \text{ (see above)} \\ y\left(\frac{1}{3}\right) &= \left(\frac{1}{3}\right)^3 - 2\left(\frac{1}{3}\right)^2 + \frac{1}{3} \\ &= \frac{1}{27} - \frac{2}{9} + \frac{1}{3} \\ &= \frac{1}{27} - \frac{6}{27} + \frac{9}{27} \\ &= \frac{4}{27} \end{aligned}$$

   There is a relative minimum at $(1, 0)$, and a relative maximum at $\left(\frac{1}{3}, \frac{4}{27}\right)$.

3. Inflection points:

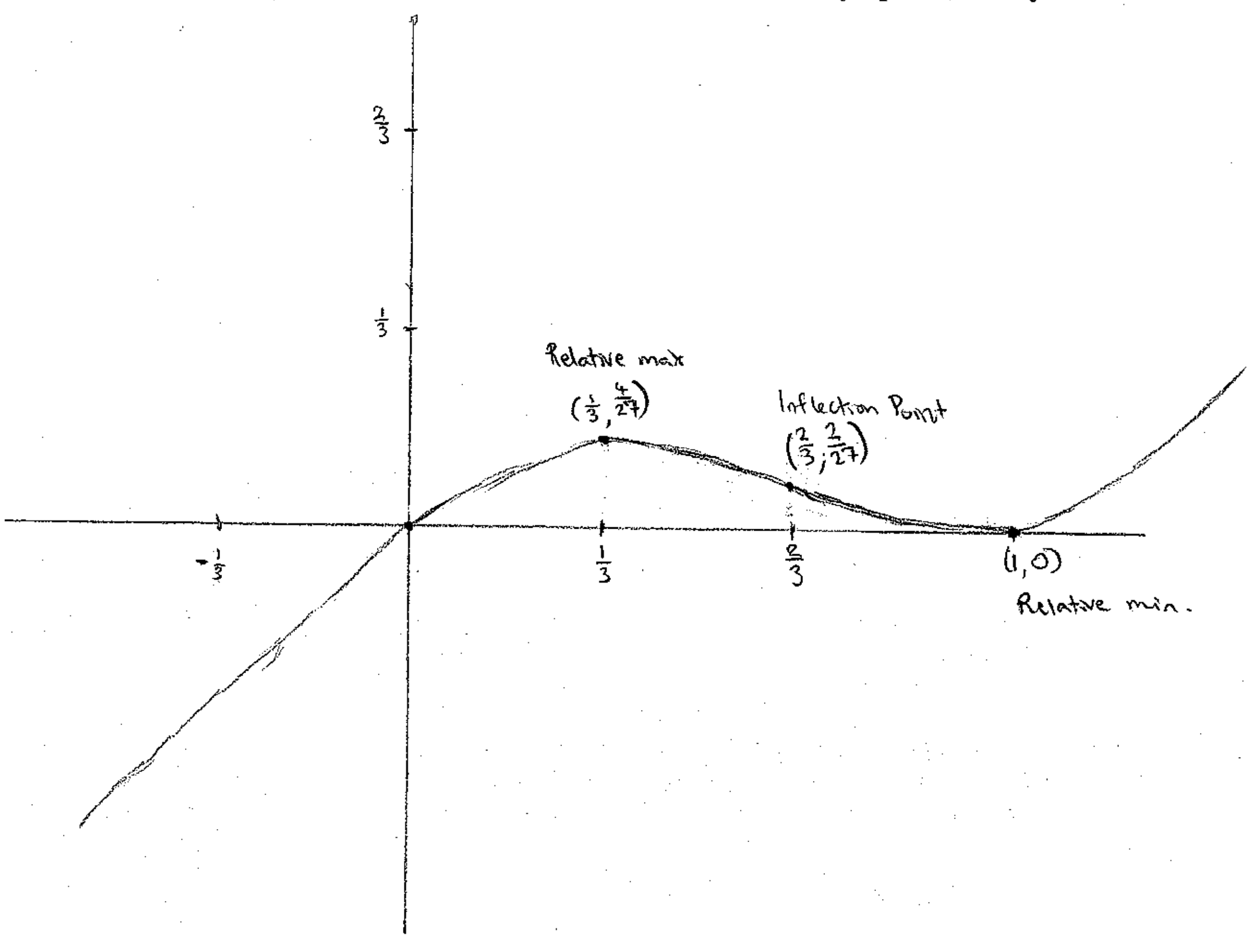
$$
\begin{aligned}
6x - 4 &= 0 \\
6x &= 4 \\
x &= \frac{2}{3}
\end{aligned}
$$

We can see that at $x = \frac{2}{3}$, y'' goes from negative to positive, so y changes from concave down to concave up. The y-value here is:

$$
\begin{aligned}
y\left(\frac{2}{3}\right) &= \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 + \frac{2}{3} \\
&= \frac{8}{27} - \frac{8}{9} + \frac{2}{3} \\
&= \frac{8}{27} - \frac{24}{27} + \frac{18}{27} \\
&= \frac{2}{27}
\end{aligned}
$$

There is an inflection point at $\left(\frac{2}{3}, \frac{2}{27}\right)$.

4. Since y is a cubic polynomial, there are no absolute extrema or asymptotes of any kind.



Problem 8 Sktch the graph of $y = xe^{-x^2}$. Clearly indicate all x-intercepts, relative and absolute extreme points, and points of inflection.

Solution:

1. x-intercepts: The only x-intercept is at $x = 0$. (e^{-x^2} is not 0 for any x.)

2. Extrema:

   (a) Derivatives:

$$\begin{aligned} y' &= 1 \cdot e^{-x^2} + x \cdot e^{-x^2} \cdot -2x \\ &= e^{-x^2}(1 - 2x^2) \\ y'' &= e^{-x^2} \cdot -2x(1 - 2x^2) + e^{-x^2}(-4x) \\ &= e^{-x^2}(-2x(1 - 2x^2) - 4x) \\ &= e^{-x^2}(-2x + 4x^3 - 4x) \\ &= e^{-x^2}(4x^3 - 6x) \end{aligned}$$

   (b) Critical points:

$$\begin{aligned} e^{-x^2}(1 - 2x^2) &= 0 \\ 1 - 2x^2 &= 0 \\ 1 &= 2x^2 \\ \frac{1}{2} &= x^2 \\ \pm\frac{1}{\sqrt{2}} &= x \end{aligned}$$

(c) Second derivative test:

$$
\begin{aligned}
y''\left(\frac{1}{\sqrt{2}}\right) &= e^{-\left(\frac{1}{\sqrt{2}}\right)^2}\left(4\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{6}{\sqrt{2}}\right) \\
&= e^{-\frac{1}{2}}\left(\frac{4}{2\sqrt{2}} - \frac{6}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{e}}\left(\frac{2}{\sqrt{2}} - \frac{6}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{e}} \cdot \frac{-4}{\sqrt{2}} \\
&= -\frac{4}{\sqrt{2e}} < 0 \\
y''\left(-\frac{1}{\sqrt{2}}\right) &= e^{-\left(-\frac{1}{\sqrt{2}}\right)^2}\left(4\left(-\frac{1}{\sqrt{2}}\right)^3 + \frac{6}{\sqrt{2}}\right) \\
&= e^{-\frac{1}{2}}\left(\frac{-4}{2\sqrt{2}} + \frac{6}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{e}}\left(\frac{-2}{\sqrt{2}} + \frac{6}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{e}} \cdot \frac{4}{\sqrt{2}} \\
&= \frac{4}{\sqrt{2e}} > 0
\end{aligned}
$$

(d) Corresponding y-values:

$$
\begin{aligned}
y\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}}e^{-\left(\frac{1}{\sqrt{2}}\right)^2} \\
&= \frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{2e}} \\
y\left(-\frac{1}{\sqrt{2}}\right) &= -\frac{1}{\sqrt{2}}e^{-\left(-\frac{1}{\sqrt{2}}\right)^2} \\
&= -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \\
&= -\frac{1}{\sqrt{2e}}
\end{aligned}
$$

There is a relative maximum at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$, and a relative minimum at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$. In the sketch below, it will be apparent that these relative extrema are, in fact, absolute.

3. Points of inflection

$$\begin{aligned}
e^{-x^2}(4x^3-6x) &= 0\\
4x^3-6x &= 0\\
2x(2x^2-3) &= 0\\
x &= 0 \text{ or}\\
2x^2-3 &= 0\\
2x^2 &= 3\\
x^2 &= \frac{3}{2}\\
x &= \pm\sqrt{\frac{3}{2}}
\end{aligned}$$

We can see that $y''>0$ for $x>\sqrt{\frac{3}{2}}$, $y''<0$ for $0<x<\sqrt{\frac{3}{2}}$, $y''>0$ for $-\sqrt{\frac{3}{2}}<x<0$, and $y''<0$ for $x<-\sqrt{\frac{3}{2}}$, so all three are inflection points. The corresponding y values are:

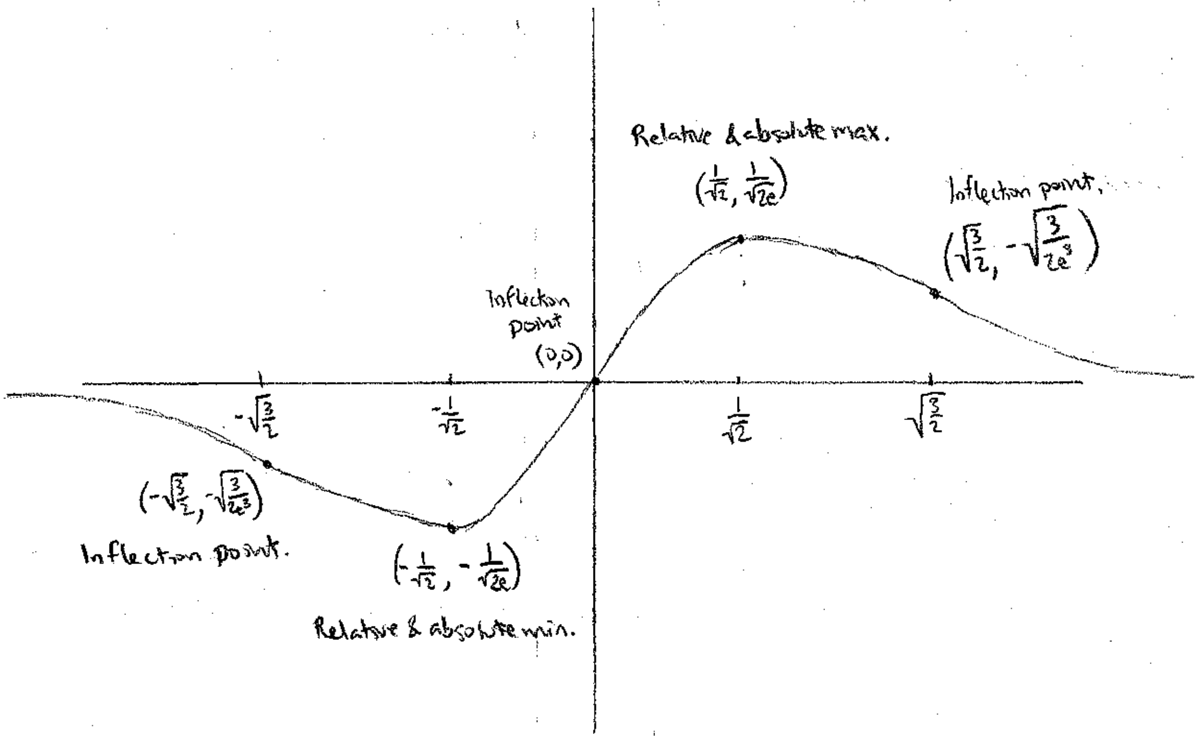
$$\begin{aligned}
y(0) &= 0\\
y\left(\sqrt{\frac{3}{2}}\right) &= \sqrt{\frac{3}{2}}e^{-\sqrt{\frac{3}{2}}^2}\\
&= \sqrt{\frac{3}{2}}e^{-\frac{3}{2}}\\
&= \sqrt{\frac{3}{2e^3}}\\
y\left(-\sqrt{\frac{3}{2}}\right) &= -\sqrt{\frac{3}{2}}e^{-\left(-\sqrt{\frac{3}{2}}\right)^2}\\
&= -\sqrt{\frac{3}{2}}e^{-\frac{3}{2}}\\
&= -\sqrt{\frac{3}{2e^3}}
\end{aligned}$$

y changes from concave down to concave up at $\left(-\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2e^3}}\right)$, from concave up to concave down at $(0,0)$, and from concave down to concave up at $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2e^3}}\right)$.

4. Asymptotes: xe^{-x^2} is never undefined, so there are no vertical asymptotes. We check for horizontal or slant asymptotes:

$$\begin{aligned}
\lim_{x\to\infty} xe^{-x^2} &= \lim_{x\to\infty}\frac{x}{e^{x^2}}\\
&= 0 \text{ (approached from above)}\\
\lim_{x\to-\infty} xe^{-x^2} &= \lim_{x\to-\infty}\frac{x}{e^{x^2}}\\
&= 0 \text{ (approached from below)}
\end{aligned}$$

The x-axis is a horizontal asymptote (on both the left and the right).



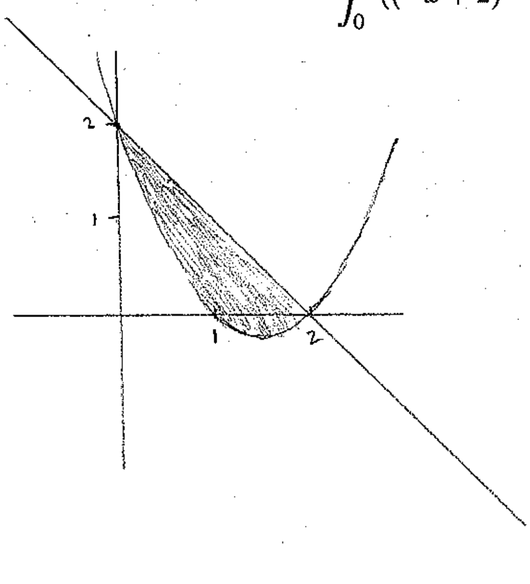
Problem 9 Find the area bounded by the curves $y = x^2 - 3x + 2$ and $y = -x + 2$.

Solution: We begin by finding the points of intersection of these curves:

$$\begin{aligned} x^2 - 3x + 2 &= -x + 2 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ x = 0 &\text{ or } x = 2 \end{aligned}$$

When $x < 0$, $-x + 2 < x^2 - 3x + 2$. When $0 < x < 2$, $x^2 - 3x + 2 < -x + 2$. When $2 < x$, $-x + 2 < x^2 - 3x + 2$. (See the sketch, below.) The area of the region enclosed by these two curves is:

$$\begin{aligned} \int_0^2 ((-x+2) - (x^2 - 3x + 2))\,\mathrm{d}x &= \int_0^2 (-x^2 + 2x)\,\mathrm{d}x \\ &= \left[-\frac{1}{3}x^3 + x^2\right]_0^2 \\ &= \left(-\frac{1}{3} \cdot 2^3 + 2^2\right) - 0 \\ &= -\frac{8}{3} + 4 \\ &= \frac{4}{3} \end{aligned}$$



Problem 10 An investor initially invests \$10,000 in a speculative venture. Suppose that the investment earns 20% interest compounded continuously for the first 5 years, and then 6% interest compounded continuously for 5 years thereafter. How much is the investment worth after 10 years?

Solution: We start by figuring out how much the investment is worth after 5 years:

$$\begin{aligned}\text{Value after 5 years} &= 10000e^{0.2\cdot 5}\\ &= 10000e\end{aligned}$$

The value after 10 years will be:

$$\begin{aligned}\text{Value after 10 years} &= (10000e)e^{0.06\cdot 5}\\ &= (10000e)e^{0.3}\\ &= 10000e^{1.3}\end{aligned}$$

Problem 11 Calculate the volume of the solid of revolution obtained by rotating the region under the graph of $y = \frac{1}{\sqrt{x}}$ between $x = 1$ and $x = 2$ about the x axis.

Solution:

$$\begin{aligned}\text{Volume} &= \int_1^2 \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx\\ &= \pi\int_1^2 \frac{1}{x}\,dx\\ &= \pi[\ln|x|]_1^2\\ &= \pi(\ln(2) - \ln(1))\\ &= \pi\cdot\ln\left(\frac{2}{1}\right)\\ &= \pi\ln 2\end{aligned}$$

Problem 12 Consider a rectangle of perimeter 12 inches. Form a cylinder by revolving this rectangle about one of its edges. What dimensions of the rectangle will result in a cylinder of maximum volume?

Solution: We imagine that this rectangle is sitting on the x-axis, with its lower lefthand corner at the origin. We rotate it around its base (i.e. around the x-axis). This cylinder we end up with is the same as if we had simply rotated the line $y = h$ (where h is the height of the rectangle) from $x = 0$ to $x = b$ (where b is the length of the base of the rectangle) around the x-axis. Therefore, its volume is given by:

$$\int_0^b \pi\cdot h^2\,dx$$

Since the perimeter of the rectangle is 12 inches, we know that $2b + 2h = 12$. Thus:

$$\begin{aligned} 2b + 2h &= 12 \\ 2h &= 12 - 2b \\ h &= 6 - b \end{aligned}$$

So the cylinder's volume is given by:

$$\begin{aligned} \text{Volume } &= \pi \int_0^b (6-b)^2 \, dx \\ &= \pi[(6-b)^2 x]_0^b \\ &= \pi((36 - 12b + b^2)b - 0) \\ &= \pi(36b - 12b^2 + b^3) \end{aligned}$$

We must now find the value of b that maximizes V:

$$\begin{aligned} \frac{dV}{db} &= \frac{d}{db}[36\pi b - 12\pi b^2 + \pi b^3] \\ &= 36\pi - 24\pi b + 3\pi b^2 \\ \frac{dV}{db} &= 0 \\ 3\pi(12 - 8b + b^2) &= 0 \\ 3\pi(b-6)(b-2) &= 0 \\ b = 6 \text{ or } b &= 2 \end{aligned}$$

We note that $\frac{dV}{db}$ goes from positive to negative at $b = 2$, and from negative back to positive at $b = 6$. So the maximum volume occurs when $b = 2$ inches. At this point, the height must be 4 inches, so the rectangle's dimensions are $2'' \times 4''$. (Alternatively, just notice that when $b = 6$, $h = 0$, so the "rectangle" is merely a line segment, leaving $b = 2$ as the only viable answer.)