1. Long Answers

(1) Let $B = \{1, 1 + x, 1 + x^2\}$ for $P_2$, the vector space of polynomials of degree at most 2. Let $T : P \to P_2$ be the linear transformation given by $T(p(x)) = p(1)x^2 - p(0)x - p(-1)$.

(a) Compute $T(x^2 - x + 2)$.

(b) Determine the matrix of $T$ with respect to the basis $B$.

(2) Suppose $A$ is a $3 \times 3$ matrix such that

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 2, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 3, and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 4.

(a) Is $A$ diagonalizable?
(b) Is $A$ invertible?
(c) Compute $A^4 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

(3) Suppose that we are given an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$, such that $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 2$, $\langle e_1, e_2 \rangle = 3$, $\langle e_1, e_3 \rangle = 4$, and $\langle e_2, e_3 \rangle = 5$, where the $e_i$ are the standard basis vectors for $\mathbb{R}^3$.

(a) Compute $\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle$ [Hint: write each vector in terms of the standard basis vectors and break up the inner product using bilinearity].

(b) What is the length of the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in the norm determined by this inner product?

(c) Write down the symmetric matrix for this inner product. That is, find the $3 \times 3$ matrix $A$ such that for all $x, y \in \mathbb{R}^3$, $\langle x, y \rangle = x^T Ay$.

(d) Find two nonzero vectors which are orthogonal with respect to this inner product.

(4) Let $W$ be the plane $x + y + z = 0$ in $\mathbb{R}^3$, and let $v = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. Compute the orthogonal projection of $v$ onto $W$. What is the distance from $W$ to $v$? [Hint: the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ form an orthogonal basis for $W$.]
2. True or False

(1) The determinant of a matrix changes sign after adding a multiple of one row to another.
False.

(2) The determinant of a matrix is nonzero if and only if the matrix is invertible.
True. See the “invertible matrix theorem”.

(3) If a vector \( \mathbf{v} \) is an eigenvector for \( A \) with eigenvalue 2 and also an eigenvector for another matrix \( B \) with eigenvalue 3, then \( \mathbf{v} \) is an eigenvector for the matrix \( A + B \) with eigenvalue 5.
True. To see this, multiply \( A + B \) against \( \mathbf{v} \) and see whether you get a multiple of \( \mathbf{v} \) - you do, namely \( 5\mathbf{v} \). Note that the fact that \( \mathbf{v} \) was a common eigenvector to both \( A \) and \( B \) was essential.

(4) There is a matrix with exactly one eigenvector.
False. If \( \mathbf{v} \) is an eigenvector for a matrix \( A \), then any nonzero multiple \( c\mathbf{v} \) of \( \mathbf{v} \) is also an eigenvector. Since \( \mathbf{v} \) is nonzero, by definition of eigenvector, these multiples are all distinct, so \( A \) actually has infinitely many distinct eigenvectors.

(5) The determinant of a matrix is equal to the product of its eigenvalues.
True. You can see this by inspecting the characteristic polynomial. You find that the constant term is equal to the determinant. But also, for any polynomial, the constant term is equal to the product of the roots. Since the roots of the characteristic polynomial are the eigenvalues, the beast has been vanquished.

(6) If \( A^2 = 0 \), then \( A \) is the zero matrix.
False. This is what your kid brother would conclude if he happened upon your linear algebra HW when he was snooping around your room. We, however, can find a counterexample thusly: take an \( n \times n \) matrix (\( n \geq 2 \)) with a 1 in the upper right corner, and zeroes everywhere else.

(7) Similar matrices have the same determinant.
True. If \( A \sim B \), then \( A = PBP^{-1} \) for some invertible \( P \), so taking determinants gives \( \det A = \det P \det B \det P^{-1} = \det B \), since \( \det P^{-1} = 1/\det P \).

(8) Any two eigenvectors of a matrix are linearly independent.
False! If \( \mathbf{v} \) is an eigenvector, then so is \( 2\mathbf{v} \). These two are not independent.

(9) The sum of the dimensions of the eigenspaces of a matrix is always less than or equal to the number of columns of the matrix.
True. Each eigenspace has dimension less than or equal to the multiplicity of the corresponding eigenvalue. The sum of the dimensions of the eigenspaces is therefore less than the sum of the multiplicities of the eigenvalues, which is the degree of the characteristic polynomial, which is also the number of columns of the matrix.

(10) If the characteristic polynomial of a matrix is \( (\lambda - 5)^3(\lambda + 1) \), then there are three linearly independent eigenvectors associated to the eigenvalue 5.
False. The matrix
\[
\begin{bmatrix}
5 & 1 & 0 & 0 \\
0 & 5 & 1 & 0 \\
0 & 0 & 5 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]
is a counterexample.

(11) If \( A^3 = A \), then 0 is an eigenvalue of \( A \).
False. The identity matrix is a counterexample.

(12) If \( \langle \cdot, \cdot \rangle \) is an inner product on \( \mathbb{R}^n \), and \( \mathbf{x}, \mathbf{y} \) are two vectors in \( \mathbb{R}^n \) such that \( \langle \mathbf{x}, \mathbf{x} \rangle = 3 \) and \( \langle \mathbf{x}, \mathbf{y} \rangle = 2 \), then \( \langle 2\mathbf{x}, \mathbf{x} + \mathbf{y} \rangle = 10 \).
True. Compute: \( \langle 2\mathbf{x}, \mathbf{x} + \mathbf{y} \rangle = 2\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle = 2 \cdot 3 + 2 \cdot 2 = 10 \).

(13) There is a real symmetric matrix whose eigenvalues are \( \pm i \).
False. Any symmetric matrix can have only real eigenvalues.

(14) If \( A \) is symmetric and \( A = PDP^{-1} \), with \( D \) diagonal, then the columns of \( P \) are orthogonal.
False. This is only the case after we choose an orthogonal basis for each eigenspace and use this orthogonal basis to construct \( P \) (or if we get lucky and each eigenspace is only one-dimensional). A counterexample is provided by taking \( A \) to be a multiple of the identity matrix. Then \( A = PDP^{-1} \) holds for \textit{any} invertible \( P \), where \( D = A \). In particular, \( P \) need not have orthogonal columns.