SYMMETRIC MATRICES AND INNER PRODUCTS

Longer (Non)Examples

(1) If A is the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, does the function $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$ define an inner product on R^2? Check the three properties of inner product.

No. Notice that the matrix is not symmetric. This suggests that the function described above might fail the symmetry property of inner products. To check this, we need to find two particular vectors $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *such that* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \neq \langle \boldsymbol{y}, \boldsymbol{x} \rangle$*. To this end, let* $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$*. Then compute* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 3$ *but* $\langle \boldsymbol{y}, \boldsymbol{x} \rangle = 4$*. So the function defined above is not symmetric and therefore does not define an inner product on* $\mathbb{R}^2$*.*

(2) The *Minkowski metric* is a function that comes up in relativity; it is "almost" an inner product on $\mathbb{R}^4$. It is given by, for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$$

So it differs from the usual dot product on $\mathbb{R}^4$ only by the presence of the minus sign in the $x_4 y_4$ term. The fourth coordinate is thought of as the time variable, whilst the other three coordinates are spatial variables. Show that the Minkowski metric is not actually an inner product on $\mathbb{R}^4$. Which property fails?

The presence of the negative sign suggests that this function might fail the positive definite property. This condition says in particular that the only vector with length zero is the zero vector. To show that this property fails, we look for a nonzero vector which, in the Minkowski metric, has length zero. Set $\boldsymbol{x} = e_1 + e_4$*. Then* $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 + 0 + 0 - 1 = 0$*, but* $\boldsymbol{x} \neq \boldsymbol{0}$*, so the Minkowski metric is not an inner product.*

(3) The real numbers form a subset of the complex numbers. Prove that a complex number z is actually a real number if and only if $z = \overline{z}$.

Write $z = x + iy$*. Then* $\overline{z} = x - iy$ *and the condition that* $z = \overline{z}$ *is equivalent to the condition that* $x + iy = x - iy$*, which is equivalent to* $y = -y$*, which in turn is equivalent to* $y = 0$*. Thus* $z = \overline{z}$ *if and only if* $y = 0$*, so* z *is real.*

(4) Why is $C[0, 1]$, the set of all continuous real-valued functions defined on the interval $[0, 2\pi]$, a vector space? In lecture it was observed that $C[0, 2\pi]$ can be given an inner product as follows: if f, g are two continuous functions on $[0, 2\pi]$, then their inner product is

$$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) dx.$$

This is certainly a function from $C[0, 2\pi] \times C[0, 2\pi]$ to $\mathbb{R}$. Check that it satisfies the three properties bilinearity, symmetry, and positive-definiteness, and thus is actually an inner product. Prove that the two functions $\sin x$ and $\cos x$ are orthogonal. Find two other functions that are orthogonal to each other. What is the largest orthogonal set of functions you can construct?

solution omitted

(5) (The Symmetrizer) If A is any matrix, let $S(A)$ be the new matrix given by the formula

$$S(A) = \frac{A + A^T}{2}$$

Show that $S(A)$ is always symmetric, no matter what A is.

Similarly, define the "antisymmetrizer" by

$$\widetilde{S}(A) = \frac{A - A^T}{2}$$

What is the relationship between $\widetilde{S}(A)$ and its transpose? [We express this relationship by saying that $\widetilde{S}(A)$ is antisymmetric.] Prove that any matrix can be decomposed into a sum of a symmetric and an antisymmetric matrix.

solution omitted

True or False

Provide reasons for the true and counterexamples for the false.

(1) Any real matrix with real eigenvalues is symmetric.
False. The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ *has real eigenvalues 1 and 2, but it is not symmetric.*

(2) A symmetric matrix is always square.
True. If A is symmetric, then $A = A^T$. If A is an $m \times n$ matrix, then its transpose is an $n \times m$ matrix, so if these are equal, we must have $m = n$.

(3) Any real matrix with real eigenvalues is similar to a symmetric matrix.
False. [The argument that follows is long, but if pressed for time, you should suspect this of being false because it's missing the condition about being orthogonally diagonalizable.] For a counterexample, let A be the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then A has real eigenvalues, namely zero, but we will show that A is not similar to any symmetric matrix. One way to do this would be to assume that it were, i.e., that $A = PSP^{-1}$ for some symmetric S, and try to derive a contradiction. This might get messy. Our lives will be easier if we remember a few facts about similar matrices: they have the same eigenvalues, and therefore the same determinant and the same trace. Knowing this, let S be any symmetric matrix - it must have the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$. Since A only has the eigenvalue 0, its trace and determinant are both zero, so for A and S to be similar we must have $\det S = ac - b^2 = 0$ and $\operatorname{Tr} A = a + c = 0$. Therefore $a = -c$ and $b^2 = ac = a(-a) = -a^2$. But $b^2 = -a^2$ if and only if both a and b are zero. But if b is zero, we've diagonalized A. We saw last week that A cannot be diagonalized - its only eigenspace is 1-D. So by assuming A is similar to S we get a contradiction, hence A is not similar to any symmetric matrix.

(4) Any two eigenvectors of a symmetric matrix are orthogonal.
False. We know only that eigenvectors from different eigenspaces are orthogonal. For a counterexample, just take any two eigenvectors, one of which is a multiple of the other.

(5) **NOTE: I made a change here - there was a typo in the earlier version**
If a symmetric matrix A has two eigenvalues λ_1, λ_2 with corresponding eigenspaces $E_1, E_2 \subset \mathbb{R}^n$ and A is diagonalizable, then $E_2 = E_1^{\perp}$.
True. Since A is diagonalizable, it has n independent eigenvectors. Therefore E_1 and E_2 together span $\mathbb{R}^n$ (this is expressed by saying that they are ***complementary*** *subspaces). Moreover, we have seen that everything in E_1 is orthogonal to everything in E_2, so the two subspaces are orthogonal. Thus each one is the orthogonal complement of the other. It is instructive to explain this by choosing an orthogonal basis for each eigenspace, and using them together to form a basis for $\mathbb{R}^n$. You should think this through on your own.*

(6) An $n \times n$ matrix A has n orthogonal eigenvectors if and only if it is symmetric.
True. It is a theorem that a matrix is symmetric if and only if it is orthogonally diagonalizable. This is the same as having n orthogonal eigenvectors since eigenvectors are nonzero and nonzero orthogonal vectors are independent.

(7) If $\mathbf{x} \in \mathbb{R}^n$, and $A = \mathbf{x}\mathbf{x}^T$, then $A^2 = A$.
False. $A^2 = (\boldsymbol{x}\boldsymbol{x}^T)(\boldsymbol{x}\boldsymbol{x}^T) = \boldsymbol{x}(\boldsymbol{x}^T\boldsymbol{x})\boldsymbol{x}^T = \boldsymbol{x}(\boldsymbol{x}\cdot\boldsymbol{x})\boldsymbol{x}^T = \boldsymbol{x}(||\boldsymbol{x}||^2)\boldsymbol{x}^T = (||\boldsymbol{x}||^2)\boldsymbol{x}\boldsymbol{x}^T = ||\boldsymbol{x}||^2 A$. This will not be equal to A unless $\boldsymbol{x}$ is a unit vector. Notice, though, that if $\boldsymbol{x}$ is a unit vector, then this matrix is the matrix for the projection along $\boldsymbol{x}$ (see question 10).

(8) If the columns of A form an orthogonal basis for $\mathbb{R}^n$, then $(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ [Hint: check the identity on the standard basis vectors $e_1, \ldots, e_n$].
False. Let the columns of A be $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. If e_i is one of the standard basis vectors, then $Ae_i = \boldsymbol{v}_i$, by

the way matrix multiplication is carried out. So if $\boldsymbol{x} = \boldsymbol{y} = e_i$, then $(A\boldsymbol{x})\cdot(A\boldsymbol{y}) = (Ae_i)\cdot(Ae_i) = \boldsymbol{v}_i\cdot\boldsymbol{v}_i$. On the other hand, $\boldsymbol{x}\cdot\boldsymbol{y} = e_i\cdot e_i = 1$. But $\boldsymbol{v}_i\cdot\boldsymbol{v}_i$ will not be equal to one unless $\boldsymbol{v}_i$ is a unit vector. So the claim is false for orthogonal columns. However, if the columns form an orthonormal basis, then the formula does hold. Such a matrix is called an ***orthogonal matrix****, and can be defined by the property above.*

(9) If A is symmetric, then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(A\mathbf{x})\cdot\mathbf{y} = \mathbf{x}\cdot(A\mathbf{y})$.
True. We proved this in discussion.

(10) If $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$, then the matrix $\mathbf{x}\mathbf{x}^T$ is the matrix for the projection onto the line spanned by $\mathbf{x}$.
False. (I should have said "orthogonal projection" to make absolutely clear what was meant) This is only true if $\boldsymbol{x}$ is a unit vector. I illustrate the difference with an example. Work in $\mathbb{R}^2$ for simplicity, and first let $\boldsymbol{x} = e_1$, the unit vector along the x-axis. Then the matrix $\boldsymbol{x}\boldsymbol{x}^T$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which is easily seen to be the projection onto the x-axis: it takes any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ to the vector $\begin{bmatrix} x \\ 0 \end{bmatrix}$. So the claim is true in this case, but only because e_1 has length 1. For if instead $\boldsymbol{x} = 2e_1$, still along the x-axis but no longer with unit length, then the matrix $\boldsymbol{x}\boldsymbol{x}^T = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, which sends any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ to the vector $\begin{bmatrix} 4x \\ 0 \end{bmatrix}$. Thus it defines a map to the x-axis, but it is not the orthogonal projection, since for example the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gets mapped to $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, but the difference $\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is not orthogonal to the x-axis (draw a picture).

(11) If a matrix M is symmetric, and $M = PDP^{-1}$, where D is diagonal, then $P^T = P^{-1}$.
False. It is true that if A is symmetric, then it is diagonalizable, so the given D must have entries the eigenvalues of A, and the columns of P must be eigenvectors of A. In fact, they are an orthogonal set of eigenvectors since A is symmetric. However, it does not follow from this that $P^T = P^{-1}$. For that formula to be true, we would need to have chosen an orthonormal *set of eigenvectors as the columns for P. I'll leave it to you to write out an explicit counterexample.*

(12) Every eigenspace of a symmetric matrix has dimension equal to the multiplicity of the corresponding eigenvalue.
True (I should have said "algebraic multiplicity" to be completely clear). This is exactly the condition for diagonlaizability: if a certain eigenvalue is repeated, say, three times in the characteristic polynomial, then that eigenspace should be three-dimensional. Since every symmetric matrix is diagonalizable, this statement is true.

(13) If A and B are symmetric, so is $A + B$.
True. Just check whether $A + B$ is equal to its own transpose, using the fact that A and B are both symmetric: $(A + B)^T = A^T + B^T = A + B$.

(14) If A and B are symmetric, then so is AB.
False. This is not true for all symmetric matrices. Interestingly, it is true whenever $AB = BA$. For a counterexample to the general case, we should look therefore for two matrices which don't commute. Try $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then A and B are both symmetric, but $AB = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, which is about as symmetric as a three-legged lizard in a Salvador Dali painting.

(15) If A is a symmetric $n \times n$ matrix and B is an $m \times n$ matrix, then $A + B^TB$ is also symmetric.
True. For any $m \times n$ matrix B, B^TB is an $n \times n$ matrix which is symmetric since $(B^TB)^T = B^T(B^T)^T = B^TB$. By question 13, then, the sum of A and B^TB is also symmetric.