SYMMETRIC MATRICES AND INNER PRODUCTS

LONGER (NON)EXAMPLES

(1) If $A$ is the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, does the function $\langle x, y \rangle = x^T A y$ define an inner product on $\mathbb{R}^2$? Check the three properties of inner product.

(2) The Minkowski metric is a function that comes up in relativity; it is “almost” an inner product on $\mathbb{R}^4$. It is given by, for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$, $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$

So it differs from the usual dot product on $\mathbb{R}^4$ only by the presence of the minus sign in the $x_4 y_4$ term. The fourth coordinate is thought of as the time variable, whilst the other three coordinates are spatial variables. Show that the Minkowski metric is not actually an inner product on $\mathbb{R}^4$. Which property fails?

(3) The real numbers form a subset of the complex numbers. Prove that a complex number $z$ is actually a real number if and only if $z = \overline{z}$.

(4) Why is $C[0,1]$, the set of all continuous real-valued functions defined on the interval $[0, 2\pi]$, a vector space? In lecture it was observed that $C[0, 2\pi]$ can be given an inner product as follows: if $f,g$ are two continuous functions on $[0, 2\pi]$, then their inner product is $\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx$.

This is certainly a function from $C[0, 2\pi] \times C[0, 2\pi]$ to $\mathbb{R}$. Check that it satisfies the three properties bilinearity, symmetry, and positive-definiteness, and thus is actually an inner product. Prove that the two functions $\sin x$ and $\cos x$ are orthogonal. Find two other functions that are orthogonal to each other. What is the largest orthogonal set of functions you can construct?

(5) (The Symmetrizer) If $A$ is any matrix, let $S(A)$ be the new matrix given by the formula $S(A) = \frac{A + A^T}{2}$

Show that $S(A)$ is always symmetric, no matter what $A$ is.

Similarly, define the “antisymmetrizer” by $\tilde{S}(A) = \frac{A - A^T}{2}$

What is the relationship between $\tilde{S}(A)$ and its transpose? [We express this relationship by saying that $\tilde{S}(A)$ is antisymmetric.] Prove that any matrix can be decomposed into a sum of a symmetric and an antisymmetric matrix.
TRUE OR FALSE

Provide reasons for the true and counterexamples for the false.

(1) Any real matrix with real eigenvalues is symmetric.
(2) A symmetric matrix is always square.
(3) Any real matrix with real eigenvalues is similar to a symmetric matrix.
(4) Any two eigenvectors of a symmetric matrix are orthogonal.
(5) If a symmetric matrix \( A \) has two eigenvalues \( \lambda_1, \lambda_2 \) with corresponding eigenspaces \( E_1, E_2 \subset \mathbb{R}^n \) and \( A \) is diagonalizable, then \( E_2 = E_1^\perp \).
(6) An \( n \times n \) matrix \( A \) has \( n \) orthogonal eigenvectors if and only if it is symmetric.
(7) If \( x \in \mathbb{R}^n \), and \( A = xx^T \), then \( A^2 = A \).
(8) If the columns of \( A \) form an orthogonal basis for \( \mathbb{R}^n \), then \( (Ax) \cdot (Ay) = x \cdot y \), for all \( x, y \in \mathbb{R}^n \) [Hint: check the identity on the standard basis vectors \( e_1, \ldots, e_n \)].
(9) If \( A \) is symmetric, then for any \( x, y \in \mathbb{R}^n \), \( (Ax) \cdot (Ay) = x \cdot (Ay) \).
(10) If \( 0 \neq x \in \mathbb{R}^n \), then the matrix \( xx^T \) is the matrix for the projection onto the line spanned by \( x \).
(11) If a matrix \( M \) is symmetric, and \( M = PDP^{-1} \), where \( D \) is diagonal, then \( P^T = P^{-1} \).
(12) Every eigenspace of a symmetric matrix has dimension equal to the multiplicity of the corresponding eigenvalue.
(13) If \( A \) and \( B \) are symmetric, so is \( A + B \).
(14) If \( A \) and \( B \) are symmetric, then so is \( AB \).
(15) If \( A \) is a symmetric \( n \times n \) matrix and \( B \) is an \( m \times n \) matrix, then \( A + B^T B \) is also symmetric.