These are problems for you to consider in small groups, and also on your own time. There will be opportunities for some of you to present solutions to the rest of the class. The asterisks indicate a very rough gauge of difficulty and/or abstraction.

1. LU-Factorization (*)

Remember or figure out the matrices corresponding to the elementary row operations. Now row reduce the following matrix to echelon form.

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \]

Call the resulting echelon matrix \( U \). For each row operation you use, write down the elementary matrix corresponding to it. Write the matrices from right to left, as a long product. Check that the product of these elementary matrices and \( A \) actually gives you the echelon matrix \( U \) you computed in the previous paragraph.

Finally, multiply just the elementary matrices together to get a matrix \( \tilde{L} \). Compute its inverse, and call this new matrix \( L \). Then \( A = LU \) is the “LU factorization” of \( A \)!

2. How does matrix multiplication affect the nullspace? (*)

Let \( A \) and \( B \) be the matrices

\[ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

What is \( \text{Null}(A) \)? Is \( \text{Null}(A) \subseteq \text{Null}(AB) \)? How about for \( BA \): is \( \text{Null}(A) \subseteq \text{Null}(BA) \)? Bonus: Any thoughts on a general principle? Are either of the above inclusions always true, regardless of the choice of \( A \) and \( B \)? Caution: one of your inclusions may be a coincidence, a result of our choice of \( A \) and \( B \). [Cf. the next problem]

3. Rank and products, abstractly (**) 

Prove that \( \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \). [Recall that to show \( a \leq \min\{b, c\} \) you must show both that \( a \leq b \) and \( a \leq c \).]

4. Application: Mathematics for airports (***)

Suppose there are \( n \) cities, and various airlines operating flights between them. If you like, you can draw a graph of the situation, with a vertex for each city, and an edge between two vertices if there is a flight from one city to the other (we assume that if you can fly from A to B, then you can also fly from B to A - this means we don’t need any arrows in our graph). Given this data, one can build a so-called “incidence matrix”, call it \( A \), whose columns and rows are indexed by the various cities, and where there is a 1 in entry \( a_{ij} \) if there is a flight from city \( i \) to city \( j \), and a zero there otherwise.

Draw an example graph for this situation with five vertices, and as many edges you like, and then write the incidence matrix for this graph. What can you say about the rows and columns of this matrix? [They tell you something about how hectic the arrivals and departures areas at the various airports are.]
Draw graphs whose incidence matrices are

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

You are to determine what information can be read off of powers of the incidence matrix. For example, if your incidence matrix was called \( A \), what does the \( i, j \)-th entry of \( A^3 \) tell you about flights with layovers?

5. All subspaces are kernels (***)

Prove that any subspace of \( \mathbb{R}^n \) is the kernel of some linear map. Give an example of such a linear map in the case where \( V \subset \mathbb{R}^3 \) is the span of \((0, 1, 1)\) and \((1, 1, 0)\).

6. Nilpotents (**)

A square matrix \( A \) is called nilpotent if there exists a positive integer \( k \) so that \( A^k = 0 \), where 0 means the zero matrix. Examples are strictly upper triangular matrices, whose only non-zero entries lie strictly above the main diagonal. Show that if \( A \) is nilpotent, then it cannot have an inverse.