SOLUTION

MATH 54 QUIZ 12

Write your name at the top. The first problem is worth four points, the second is worth two, and the T/F are worth one each.

(1) Find the Fourier series for the function \( f(x) = \begin{cases} 2 & -2 < x < 0 \\ -2 & 0 < x < 2 \end{cases} \) on the interval \(-2 < x < 2\).

\[ f \quad \text{is odd so} \quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \]

where \[ b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin\left(\frac{n\pi x}{2}\right) dx = \int_{0}^{2} (-2) \sin\left(\frac{n\pi x}{2}\right) dx \]

\[ = \frac{4}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \bigg|_{x=0}^{x=2} = \begin{cases} 0 & n \text{ even} \\ \frac{8}{n\pi} & n \text{ odd} \end{cases} \]

\[ \Rightarrow \quad f(x) \sim \sum_{k=0}^{\infty} \frac{8}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi x}{2}\right) \]

(2) Show that 1 and \( \sin x \) are orthogonal with respect to the inner product \( (f, g) = \int_{-\pi}^{\pi} f(x) g(x) dx \) on the space of piecewise continuous functions on \((-\pi, \pi)\).

\[ \langle 1, \sin x \rangle = \int_{-\pi}^{\pi} 1 \cdot \sin x \, dx = -\cos x \bigg|_{-\pi}^{\pi} = 0 \]
TRUE OR FALSE

(1) The function \( f(x) = x \sin x \) is even.
(2) If \( f(x) \) is a piecewise continuous function on the interval \((-T, T)\), and if 
\( \hat{f}(x) \) denotes the fourier series for \( f \) on this interval, then \( f(x) = \hat{f}(x) \) for every \( x \) in \((-T, T)\).
(3) A constant function is periodic.
(4) In solving the heat equation \( u_t = \beta u_{xx} \) subject to the inhomogeneous boundary conditions \( u(0, t) = 1, u(L, t) = 3 \), we should try a solution of the form \( u(x, t) = w(x, t) + v(x) \); where \( w \) and its derivatives tend to zero as \( t \) goes to infinity.

(1) True: \( x \) and \( \sin x \) are both odd, and
the product of two odd functions is an even function (they're not like numbers!)
(Or, just compute: \( \int (-x) \sin (-x) \, dx = x \sin x = f(x) \)
\( \Rightarrow \hat{f}(x) \) even)

(2) False: If \( f \) is discontinuous, then
\( f \) and its Fourier series \( \hat{f} \) might not agree at the points of discontinuity.
A counterexample is problem 80:
There \( f(0) = 2 \), but \( \hat{f}(0) = \sum_{k=0}^{\infty} (-1)^k \sin(k) = 0 \)

(3) True. \( f \) is periodic if
\( f(x+T) = f(x) \) for some \( T \) (and for all \( x \))
If \( f(x) = k \), a constant, then \( f(x+T) = k \), too.

(4) True. Try it for yourself!