SOLUTIONS TO MATH 54 SAMPLE MIDTERM 1

Solution 1.  
(1) False. One counterexample is \( \vec{x} = [0, 1]^T \).
(2) True. The number of free variables of this system is the dimension of the nullspace of \( A \), and a vector space contains nontrivial (i.e., nonzero) elements if and only if its dimension is at least one.
(3) False. The equation \( A\vec{x} = \vec{0} \) always admits the trivial solution \( \vec{x} = \vec{0} \), regardless of \( A \).
(4) False. One nontrivial dependence relation among these vectors is \( [1, 2]^T - 2 \cdot [2, 3]^T + [3, 4]^T = [0, 0]^T \).
(5) False. The collection \( \{[1, 0]^T, [2, 0]^T, [0, 1]^T\} \) is linearly dependent, but \( [0, 1]^T \) is not a linear combination of the other two vectors. However it is true that some vector in the collection must be a linear combination of the others.
(6) True. A vector can be written as \( A\vec{x} \) if and only if it is a linear combination of the columns of \( A \) (with weights given by the entries of \( \vec{x} \)).

Solution 2.  
(1) True. The dimension of the column space is the number of pivot columns of \( A \) and the dimension of the nullspace is the number of non-pivot columns of \( A \), so they add up to the total number of columns of \( A \), namely 3.
(2) True. This is the definition of a linear transformation.
(3) False. There are many other possibilities, including reflections, shears, dilations, projections, etc.
(4) True. If the equation \( A\vec{x} = \vec{0} \) has no free variables then \( A \) has a pivot in every column, so because \( A \) is square, \( \text{rref}(A) \) must be the identity.
(5) False. They span a two-dimensional subspace of \( \mathbb{R}^3 \).
(6) False. The determinant of an invertible matrix can be any scalar except zero.

Solution 3.  
\[
AB = \begin{bmatrix} 5 & 10 \\ 2 & 32 \end{bmatrix} \quad BA = \begin{bmatrix} 10 & 9 & 8 \\ 8 & 10 & 12 \\ 9 & 13 & 17 \end{bmatrix} \quad BC = \begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}
\]

\( AC \) is not defined because the number of columns of \( A \) is different from the number of rows of \( C \).

Solution 4. It is in the span because the matrix
\[
\begin{bmatrix}
1 & -1 & 8 & 4 \\
1 & 0 & 8 & 3 \\
2 & -1 & 6 & 2 \\
2 & 3 & 12 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0.5 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

has no pivot in the last column. You don’t have to go beyond row echelon form to see this, but the reduced row echelon form tells you weights that express \([4, 3, 2, 1]\) as a linear combination of the other vectors.
Solution 5. We have 
\[
\begin{bmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
2 & 3 & 4 & 0 & 1 & 0 \\
3 & 4 & 4 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -4 & 4 & -1 \\
0 & 1 & 0 & 4 & -5 & 2 \\
0 & 0 & 1 & -1 & 2 & -1
\end{bmatrix},
\]
So \( A \) is invertible and
\[
A^{-1} = \begin{bmatrix}
-4 & 4 & -1 \\
4 & -5 & 2 \\
-1 & 2 & -1
\end{bmatrix}.
\]
The matrix \( B \) is not invertible because its columns are linearly dependent (the last two are equal.)

Solution 6. We have 
\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 1 & 0 & 0 \\
2 & 3 & 4 & 5 & 0 & 1 & 0 \\
3 & 4 & 5 & 6 & 0 & 0 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -1 & -2 & 0 & -4 & 3 \\
0 & 1 & 2 & 3 & 0 & 3 & -2 \\
0 & 0 & 0 & 0 & 1 & -2 & 1
\end{bmatrix}.
\]
A basis for the row space of \( A \) is given by the nonzero rows of \( \text{rref}(A) \), namely
\(([1, 0, -1, -2], [0, 1, 2, 3])\).

A basis for the column space of \( A \) is given by the pivot columns of \( A \), namely
\(([1, 2, 3]^T, [2, 3, 4]^T)\).

A basis for the nullspace of \( A \) is given by the fundamental solutions the equation \( A\vec{x} = \vec{0} \), or equivalently to \( \text{rref}(A)\vec{x} = \vec{0} \); these can be obtained by setting the pair of free variables \( x_2, x_3 \) to \( 1, 0 \) and then to \( 0, 1 \) and solving for \( x_1 \) and \( x_2 \), yielding
\(([1, -2, 1, 0]^T, [2, -3, 0, 1]^T)\).

A basis for the left nullspace of \( A \) is given by the portion of the rows of \( \text{rref}([A \ I_3]) \) extending the zero rows of \( \text{rref}(A) \), namely
\(([1, -2, 1]^T)\).

You can use a row echelon form of \( A \) other than \( \text{rref}(A) \), but you won’t get a Schubert basis for the row space, and it will be a little harder to find the fundamental solutions to \( A\vec{x} = \vec{0} \).

Solution 7. Notice that there cannot be a zero row in \( \text{rref}(A) \). (If there were, we could find a vector \( \vec{b} \) such that \([A \ \vec{b}]\) had a pivot in the last column, making the equation \( A\vec{x} = \vec{b} \) unsolvable.) For any vector \( \vec{b} \), the solution set of \( A\vec{x} = \vec{b} \) is the same as the solution set of the system whose augmented matrix is \( \text{rref}([A \ \vec{b}]) \). But by our first observation and the fact that \( A \) is square, we must have \( \text{rref}(A) = I_n \), so this solution set consists of the single vector which forms the rightmost column of \( \text{rref}([A \ \vec{b}]) \).