1. Computations

Solve the initial boundary value problem

\[ u_t = 9u_{xx} + e^x \quad 0 < x < 2, \quad t > 0 \]
\[ u(0, t) = u(2, t) = 0 \]
\[ u(x, 0) = \sin 2\pi x - 2 \sin \frac{5\pi x}{2} \]

Solution: This is the heat equation with an external heat source term \( e^x \), and homogeneous boundary conditions \( u(0, t) = u(2, t) = 0 \). The presence of the external heat source \( e^x \) causes us to look for solutions of the form

\[ u(x, t) = w(x, t) + v(x) \]

where \( w \) and its partials go to zero as \( t \) goes to infinity. If we plug in this expression to the heat equation, we get

\[ w_t(x, t) = 9w_{xx}(x, t) + 9v''(x) + e^x \quad (\ast) \]

If we let \( t \to \infty \), the terms involving \( w \) drop out, giving the equation \( 0 = 9v''(x) + e^x \), which we use to solve for \( v \) by integrating twice:

\[
\begin{align*}
v''(x) &= -\frac{1}{9}e^x \\
v'(x) &= -\frac{1}{9}e^x + C \\
v(x) &= -\frac{1}{9}e^x + Cx + D
\end{align*}
\]

Now we use the boundary conditions to solve for the constants \( C \) and \( D \). The given boundary conditions \( u(0, t) = u(2, t) = 0 \) imply that \( v(0) = v(2) = 0 \), so we get:

\[
\begin{align*}
v(0) &= -\frac{1}{9} + D = 0 \\
v(2) &= -\frac{1}{9}e^2 + 2C + D = 0
\end{align*}
\]

Solving these two equations gives \( D = \frac{1}{9} \) and \( C = \frac{1}{18}(e^2 - 1) \). Thus we have found that

\[
v(x) = -\frac{1}{9}e^x + \frac{1}{18}(e^2 - 1)x + \frac{1}{9}.
\]

Now if we plug this back into the equation \((\ast)\) above, we get the simpler equation

\[ w_t(x, t) = 9w_{xx}(x, t). \]

with the same homogeneous boundary conditions, but with the new initial condition \( w(x, 0) = \sin 2\pi x - 2 \sin \frac{5\pi x}{2} - v(x) \). This is your run-of-the-mill heat equation. We have seen many times that there are solutions, for each \( n \), of the form

\[ w_n(x, t) = b_ne^{-3(\frac{\pi n}{2})^2t} \sin \left( \frac{n\pi x}{2} \right). \]
and the general solution is obtained as an infinite sum of functions of this form.

The $b_n$s are determined by taking $t = 0$ and using the initial condition, which amounts to finding the Fourier sine series for $f(x) = \sin 2\pi x - 2\sin \frac{5\pi x}{2} - v(x) = \sin 2\pi x - 2\sin \frac{5\pi x}{2} + \frac{1}{9} e^x - \frac{1}{18} (e^2 - 1)x - \frac{1}{9}.$

To do this, we work separately on the three functions $\sin 2\pi x - 2\sin \frac{5\pi x}{2}, \frac{1}{9} e^x,$ and $-\frac{1}{18} (e^2 - 1)x - \frac{1}{9}.$ The first function is already a Fourier sine series on $[0, 2]$: its Fourier sine coefficients, call them $c_n$, are $c_4 = 1, c_5 = -2,$ and all others are zero.

The Fourier sine coefficients $d_n$ of the function $\frac{1}{9} e^x$ are found by

$$d_n = \frac{2}{2} \cdot \frac{1}{9} \int_0^2 e^x \sin \frac{n\pi x}{2} dx = \frac{1}{9} \left[ e^x \sin \frac{n\pi x}{2} \right]_0^2 - \frac{n\pi}{2} \cdot \frac{1}{9} \int_0^2 e^x \cos \frac{n\pi x}{2} dx$$

$$= 0 - \frac{n\pi}{2} \cdot \frac{1}{9} \left[ \left( e^x \cos \frac{n\pi x}{2} \right)_0^2 + \frac{n\pi}{2} \int_0^2 e^x \sin \frac{n\pi x}{2} dx \right]$$

the integral on the right is the original integral, so we add it back over to the left side, giving

$$\left(1 + \left(\frac{n\pi}{2}\right)^2\right) \frac{1}{9} \int_0^2 e^x \sin \frac{n\pi x}{2} dx = -\frac{n\pi}{2} \cdot \frac{1}{9} \cdot \left( e^2(-1)^n - 1 \right),$$

so

$$d_n = \frac{n\pi (1 - e^2(-1)^n)}{18(1 + (\frac{n\pi}{2})^2)}$$

Next we compute the Fourier sine coefficients $e_n$ of the third function, $-\frac{1}{18} (e^2 - 1)x - \frac{1}{9}.$ This function is linear, and the Fourier coefficients can be computed using the tabular method for repeated integration by parts, as done in the solution to the previous worksheet. If we write our function as $Ax + B$ for simplicity, we get

$$e_n = -\frac{n\pi}{2} \cdot [2A + B](-1)^n - B$$

Plugging in $A = -\frac{1}{18} (e^2 - 1)$ and $B = -\frac{4}{9},$ we get

$$e_n = -\frac{n\pi}{2} \cdot \left( -\frac{1}{9} (e^2 - 1) - \frac{1}{9} \right)(-1)^n + \frac{1}{9} \right) = \frac{n\pi}{18} \cdot (e^2(-1)^n + 1)$$

The Fourier coefficients $b_n$ of our original function are then just given by the sum $b_n = c_n + d_n + e_n$ of the Fourier coefficients of these three constituent functions. To obtain the final answer, one substitutes these horrendous terms $b_n$ into

$$w_n(x, t) = b_n e^{-3t(\frac{n\pi x}{2})^2} \sin \left( \frac{n\pi x}{2} \right)$$

Now that we’ve found $w$ and $v$, we can build our original solution $u$, by adding these two functions. Since the Fourier coefficients turned out so awful, we don’t write them down, but in nice cases you would obtain an expression for $u(x, t)$ which was some somehow of sine functions plus the extra terms coming from the function $v(x)$

**PLEASE NOTE** - these integrals turned out awful. Sorry. Do not expect anything quite so elaborate on the exam (but don’t expect a walk in the park, either).
2. Linear Algebra

(1) An \( n \times n \) matrix \( A \) is diagonalizable if

(a) It has \( n \) distinct eigenvectors.
(b) There is a matrix \( P \) and a diagonal matrix \( D \) such that \( AP = PD \)
(c) It has \( n \) distinct eigenvalues
(d) It preserves lengths and angles.

Solution: (a) is false - every matrix has infinitely many distinct eigenvectors. (b) is false, since we could take \( P \) to be the zero matrix - then the given equation is true regardless of \( A \) and \( D \). If I had required \( P \) to be invertible, then this would be the definition of diagonalizability of \( A \). (c) is true, but remember that this isn’t the ONLY way a matrix could be diagonalizable, i.e., there are some matrices which have repeated eigenvalues, but are nonetheless diagonalizable. Finally, (d) is a property of orthogonal transformations, not diagonalizable matrices, so that one’s out.

(2) Which of the following matrices is not diagonalizable (over \( \mathbb{R} \))?

(a) \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
1 & 1 \\
0 & 2
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & -1 \\
2 & 0 & 1
\end{bmatrix}
\]
(d) \[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Solution: There was a typo here - sorry!!! (c) is diagonalizable over \( \mathbb{C} \), but not over \( \mathbb{R} \), so (c) is correct, but also (d) is correct - (d) has the repeated eigenvalue \( \lambda = 1 \), with multiplicity three, but if you look in the nullspace of \( A - \lambda I \) (where \( A \) stands for the given matrix), you will find at most two independent vectors. So there are not enough independent eigenvectors, hence \( A \) is not diagonalizable (over \( \mathbb{R} \) or \( \mathbb{C} \), in fact!). Sincere apologies if you lost sleep over this. Hopefully you know me well enough by now to realize that there was a typo!