Write your name at the top. There are 30 points total.

(1) (a) [4 pts] Compute the double integral $\int_0^3 \int_0^1 (1 + 4xy) \, dx \, dy$

\[ \int_0^3 \left[ x + 2x^2 y \right]_0^1 \, dy = \int_0^3 (1 + 2y) \, dy = y + y^2 \bigg|_0^3 \]

\[ = 12 - 0 = 12 \]

(b) [4 pts] If

\[ \iint_R f(x, y) \, dA = \int_{-2}^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} f(x, y) \, dy \, dx, \]

sketch the region $R$.

(2) (a) [3 pts] The function $T(x, y)$ represents the temperature at the point $(x, y)$ on a hot metal plate. A bug has wandered onto this plate, and is at the point $(1, 3)$. She quickly calculates that $\nabla T(1, 3) = (-1, 1)$. In what direction should she begin walking to cool off the fastest? Should she necessarily continue walking in this direction to cool off the fastest?

"In direction $\langle -1, 1 \rangle$. No - the gradient may change direction as she moves."

(b) [3 pts] If $f$ is a differentiable function on $\mathbb{R}^2$, and $(a, b)$ a point in $\mathbb{R}^2$ such that $D_u(a, b) = 0$ for all unit vectors $u$, what can you say about $\nabla f(a, b)$?

\[ \nabla f(a, b) = \langle 0, 0 \rangle \]

(c) [3 pts] Can the tangent plane to the surface $F(x, y, z) = xz + \sin(xy) = 1$ ever be parallel to $yz$-plane?

"No" because

\[ \nabla F(x, y, z) = \langle z + y \cos xy, x \cos xy, 1 \rangle \]

Since $z = 0 \Rightarrow x \cos xy = 0 \Rightarrow \langle 0, 0, 1 \rangle = \langle 0, 0, 1 \rangle$
(3) [6 pts] The function \( f(x, y) = x^4 + y^4 - 4xy + 2 \) has three critical points: \((-1, -1), (0, 0), \) and \((1, 1)\). Classify them as local maxima, local minima or saddle points.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4x^3 - 4y \\
\frac{\partial f}{\partial y} &= 4y^3 - 4x \\
\frac{\partial^2 f}{\partial x^2} &= 12x^2 \\
\frac{\partial^2 f}{\partial x \partial y} &= -4 \\
\frac{\partial^2 f}{\partial y^2} &= 12y^2
\end{align*}
\]

\[
\Delta(x, y) = 144x^2y^2 - 16
\]

\[
\Delta(1, 1) = \Delta(-1, -1) = 128 > 0
\]

\[
f_{xx}(1, 1) = f_{xx}(-1, -1) = 12 > 0
\]

\[
\Rightarrow (1, 1) \text{ and } (-1, -1) \text{ are local minima}
\]

\[
N(0, 0) = -16 < 0
\]

\[
\Rightarrow (0, 0) \text{ is a saddle point}
\]

(4) [7 pts] Find the maximum distance from the curve \( 4x^2 + xy + 4y^2 = 1 \) to the origin in \( \mathbb{R}^2 \).

You may assume that there is a maximum.

Maximize

\[
\begin{align*}
\frac{\partial f}{\partial x} &= x^2 + y^2 \quad \text{subject to } g(x, y) = 4x^2 + xy + 4y^2 = 1
\end{align*}
\]

\[
\begin{align*}
f_x &= 2x = (8x + y)\lambda = \lambda g_x \\
f_y &= 2y = (8y + x)\lambda = \lambda g_y \\
4x^2 + xy + 4y^2 &= 1
\end{align*}
\]

Note \( \lambda \neq 0 \), or else \( x = y = 0 \) \( \Rightarrow \) not on curve.

If \( y = 0 \), \( 8y + x = x = 0 \), since \( x \neq 0 \), so \( y \neq 0 \) either.

\[
\frac{x}{y} = \frac{8x + y}{8y + x} \Rightarrow 8y + x = 8xy + y^2 \Leftrightarrow x^2 = y^2
\]

Hence \( \frac{x}{y} = \frac{8x + y}{8y + x} \).

So \( x = \pm y \).

If \( x = y \), we have \( 9x^2 = 1 \) so \( x = y = \pm \frac{1}{3} \).

If \( x = -y \), we have \( 7x^2 = 1 \) so \( x = \pm \frac{\sqrt{7}}{7} = -y \).

Compute \( f \) at these four points:

\[
\begin{align*}
f\left(\frac{1}{3}, \frac{1}{3}\right) &= f\left(-\frac{1}{3}, -\frac{1}{3}\right) = \frac{2}{9} \\
f\left(\frac{\sqrt{7}}{7}, \frac{-\sqrt{7}}{7}\right) &= f\left(-\frac{\sqrt{7}}{7}, -\frac{\sqrt{7}}{7}\right) = \frac{2}{7}
\end{align*}
\]

So max distance is \( \frac{2}{7} \).