1. Stewart 16.1.36

[5 pts]

(a) Sketch the vector field \( \mathbf{F}(x,y) = \mathbf{i} + x\mathbf{j} \) and then sketch some flow lines (lines whose velocity vectors are given by the values of \( \mathbf{F} \) at the points along the curve). What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are \( x = x(t), y = y(t) \), what differential equations do these functions satisfy? Deduce that \( \frac{dy}{dx} = x \).

(c) If a particle starts at the origin in the velocity field given by \( \mathbf{F} \), find an equation of the path it follows.

Solution:

(a) The vector field looks like this:

and the flow lines look like parabolas.

(b) By the definition of flow lines, the tangent vector to these parametric curves at each point must equal the value of \( \mathbf{F} \) at each point. Thus \((x',y') = (1,x)\), so the two functions \( x(t) \) and \( y(t) \) must satisfy

\[
x'(t) = 1 \quad \text{and} \quad y'(t) = x(t)
\]

Since \( \frac{dy}{dx} = \frac{y'(t)}{x'(t)} \), we have \( \frac{dy}{dx} = x \).

(c) To find the equation of the flow line which passes through the origin, we solve the equation \( \frac{dy}{dx} = x \), giving \( y = x^2 + C \), and the condition that it must pass the origin forces \( C = 0 \).

2. Stewart 16.2.18

[5 pts] The figure shows a vector field \( \mathbf{F} \) and two curves \( C_1 \) and \( C_2 \). Are the line integrals of \( \mathbf{F} \) over \( C_1 \) and \( C_2 \) positive, negative, or zero? Explain.
**Solution:** The angle between $\mathbf{F}$ and $C_1$ is always between zero and 90°, so the work done along $C_1$ is positive. The work done along (roughly) the first half of $C_2$ is positive, and along the second half, negative. However, since the field is stronger along the second half, the overall work done will be negative.

3. **Stewart 16.2.42**

**[5 pts]** The force exerted by an electric charge at the origin on a charged particle at a point $(x, y, z)$ with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K \mathbf{r}/|\mathbf{r}|^3$, where $K$ is a constant. Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

**Solution:** The straight line path is parametrized by $(1 - t)(2, 0, 0) + t(2, 1, 5) = \langle 2, t, 5t \rangle$, with tangent vector $\langle 0, 1, 5 \rangle$. The line integral for the work done is

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \frac{K}{(4 + t^2 + 25t^2)^{3/2}} (2, 5t) \cdot \langle 0, 1, 5 \rangle dt
\]

\[
= \int_0^1 \frac{26Kt}{(4 + 26t^2)^{3/2}} dt
\]

\[
= \frac{K}{2} \int_4^{30} \frac{1}{u^{3/2}} du, \text{ where } u = 4 + 26t^2
\]

\[
= -K \left[ \frac{1}{\sqrt{u}} \right]_4^{30}
\]

\[
= K \left( \frac{1}{2} - \frac{1}{\sqrt{30}} \right)
\]

Note that the work is positive, which makes sense since the particle is moving further away from the origin, and the electric field points radially outward.