ASSIGNMENT 6 SOLUTION

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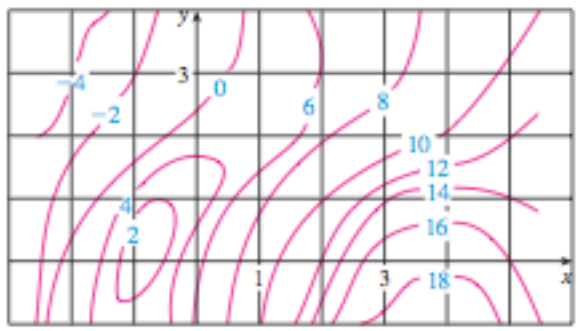
1. Stewart 14.3.10

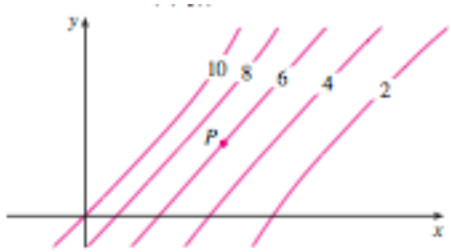
[3 pts]
A contour map for a function f is given. Use it to estimate $f_x(2,1)$ and $f_y(2,1)$.



Solution: We can estimate f_x by observing that as x goes from 1.5 to 2.5, with $y = 1$ fixed, f seems to go from about 9 to 12. Therefore $f_x(2,1)$ is approximately 3. As y goes from 0.5 to 1.5, with $x = 2$ fixed, f goes from about 11 to about 9, so $f_y(2,1)$ is roughly -2.

2. Stewart 14.3.70

[3 pts] *Level curves for a function f are shown. Determine whether the following partial derivatives are positive or negative at the point P: (a) f_x (b) f_y (c) f_{xx} (d) f_{xy} (e) f_{yy}.*



Solution:

(a) $f_x < 0$
(b) $f_y > 0$
(c) $f_{xx} > 0$, since the x-slope becomes less negative as we move to the right.
(d) $f_{xy} < 0$, since the x-slope becomes more negative as we move upwards.
(e) $f_{yy} > 0$, since the y-slope becomes more positive as we move upwards.

3. Stewart 14.3.94

[3 pts] *If $f(x,y) = \sqrt[3]{x^3+y^3}$, find $f_x(0,0)$.*

Solution: The expression $f_x(x,y) = \frac{x^2}{(x^3+y^3)^{2/3}}$ is not valid at the origin, so we must use the limit definition of the partial derivative:

$$f_x(0,0) = \lim_{h\to 0} \frac{\sqrt[3]{(0+h)^3+0^3} - \sqrt[3]{0^3+0^3}}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

4. Stewart 14.4.6

[3 pts] *Find an equation of the tangent plane to the surface* $z = e^{x^2-y^2}$ *at the point* $P = (1, -1, 1)$.
Solution: The normal vector to the tangent plane is

$$\mathbf{n} = \langle f_x(1,-1), f_y(1,-1), -1\rangle = \langle 2, 2, -1\rangle$$

Thus the equation for the plane is $\langle 2, 2, -1\rangle \cdot \langle x-1, y+1, z-1\rangle = 0$, which reduces to

$$2x + 2y - z = -1.$$

5. Stewart 14.5.53

[3 pts] *If* $z = f(x, y)$, *where* $x = r\cos\theta$ *and* $y = r\sin\theta$, *show that*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

Solution: The key ingredients in the right hand side are as follows

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \frac{\partial^2 z}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta \\
\frac{\partial^2 z}{\partial \theta^2} &= \frac{\partial^2 z}{\partial x^2} r^2\sin^2\theta + \frac{\partial^2 z}{\partial y \partial x}(-r^2\cos\theta\sin\theta) + \frac{\partial z}{\partial x}(-r\cos\theta) \\
&\quad + \frac{\partial^2 z}{\partial x \partial y}(-r^2\sin\theta\cos\theta) + \frac{\partial^2 z}{\partial y^2}(r^2\cos^2\theta) + \frac{\partial z}{\partial y}(-r\sin\theta) \\
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta
\end{aligned}$$

Putting these all into the right hand side and simplifying reduces to the left hand side.