ASSIGNMENT 6 SOLUTION

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1. Stewart 14.3.10

[3 pts]
A contour map for a function $f$ is given. Use it to estimate $f_x(2,1)$ and $f_y(2,1)$.

Solution: We can estimate $f_x$ by observing that as $x$ goes from 1.5 to 2.5, with $y = 1$ fixed, $f$ seems to go from about 9 to 12. Therefore $f_x(2,1)$ is approximately 3. As $y$ goes from 0.5 to 1.5, with $x = 2$ fixed, $f$ goes from about 11 to about 9, so $f_y(2,1)$ is roughly -2.

2. Stewart 14.3.70

[3 pts] Level curves for a function $f$ are shown. Determine whether the following partial derivatives are positive or negative at the point $P$: (a) $f_x$ (b) $f_y$ (c) $f_{xx}$ (d) $f_{xy}$ (e) $f_{yy}$.

Solution:
(a) $f_x < 0$
(b) $f_y > 0$
(c) $f_{xx} > 0$, since the $x$-slope becomes less negative as we move to the right.
(d) $f_{xy} < 0$, since the $x$-slope becomes more negative as we move upwards.
(e) $f_{yy} > 0$, since the $y$-slope becomes more positive as we move upwards.

3. Stewart 14.3.94

[3 pts] If $f(x,y) = \sqrt[3]{x^3 + y^3}$, find $f_x(0,0)$.

Solution: The expression $f_x(x,y) = \frac{x^2}{(x^3+y^3)^{2/3}}$ is not valid at the origin, so we must use the limit definition of the partial derivative:

$$f_x(0,0) = \lim_{h \to 0} \frac{\sqrt[3]{(0+h)^3 + 0^3} - \sqrt[3]{0^3 + 0^3}}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$
4. Stewart 14.4.6

[3 pts] Find an equation of the tangent plane to the surface \( z = e^{x^2-y^2} \) at the point \( P = (1, -1, 1) \).

**Solution:** The normal vector to the tangent plane is

\[ n = \langle f_x(1, -1), f_y(1, -1), -1 \rangle = \langle 2, 2, -1 \rangle \]

Thus the equation for the plane is \( 2(x - 1) + 2(y + 1) - (z - 1) = 0 \), which reduces to

\[ 2x + 2y - z = -1. \]

5. Stewart 14.5.53

[3 pts] If \( z = f(x, y) \), where \( x = r \cos \theta \) and \( y = r \sin \theta \), show that

\[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r} \frac{\partial z}{\partial r} \]

**Solution:** The key ingredients in the right hand side are as follows

\[ \frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2} \cos^2 \theta + \frac{\partial^2 z}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 z}{\partial y \partial x} \sin \theta \cos \theta + \frac{\partial^2 z}{\partial y^2} \sin^2 \theta \]

\[ \frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} r^2 \sin^2 \theta + \frac{\partial^2 z}{\partial x \partial y} (-r^2 \cos \theta \sin \theta) + \frac{\partial z}{\partial x} (-r \cos \theta) \]

\[ + \frac{\partial^2 z}{\partial x \partial y} (-r^2 \sin \theta \cos \theta) + \frac{\partial^2 z}{\partial y^2} (r^2 \cos^2 \theta) + \frac{\partial z}{\partial y} (-r \sin \theta) \]

\[ \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta \]

Putting these all into the right hand side and simplifying reduces to the left hand side.