ASSIGNMENT 4 SOLUTION

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1. Stewart 12.3.26

[2 pts] *For what values of b are the vectors $\langle -6, b, 2\rangle$ and $\langle b, b^2, b\rangle$ orthogonal?*

Solution: They are orthogonal if and only if

$$\langle -6, b, 2\rangle \cdot \langle b, b^2, b\rangle = 0,$$

which happens when $-6b + b^3 + 2b = 0$. The solutions of this equation are $b = -2, 0, 2$.

2. Stewart 12.3.44(b)

[2pts]

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be nonzero vectors. Under what circumstances is the (vector) projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ equal to that of $\boldsymbol{b}$ onto $\boldsymbol{a}$?

Solution: This will be true when

$$\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}.$$

If $\mathbf{a}\cdot\mathbf{a} = 0$, then both sides of the equation are zero. So one circumstance in which the two projections agree is when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, in which case both projections are zero. Otherwise, we may cancel $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{a}$ to give

$$\frac{1}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}.$$

This is an equality between two vectors. Therefore $\mathbf{a}$ and $\mathbf{b}$ must have the same direction. The lengths of both sides must also be equal, which forces $|\mathbf{a}| = |\mathbf{b}|$. Thus the only other possibility is that $\mathbf{a} = \mathbf{b}$.

3. Stewart 12.3.53

[2 pts]

A molecule of methane, CH_4, is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The bond angle is the angle formed by the $H - C - H$ combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about $109.5°$. [Hint: Take the vertices of the tetrahedron to be the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, and $(1, 1, 1)$. Then the centroid is $(1/2, 1/2, 1/2)$].

Solution: To compute the angle between two of the bonds, we need to write the bonds as vectors. This is done by taking the vectors whose tail is at the carbon atom in the center, and whose head is at each of the hydrogen atoms. Using the first two hydrogen atoms listed in the hint, we get two vectors $\mathbf{a}_1 = \langle 1/2, -1/2, -1/2\rangle$ and $\mathbf{a}_2 = \langle -1/2, 1/2, -1/2\rangle$. Using the cosine formula for the dot product we get

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = |\mathbf{a}_1||\mathbf{a}_2|\cos\theta,$$

where θ is the bond angle. Plugging in for $\mathbf{a}_1$ and $\mathbf{a}_2$ and solving for θ between 0 and $180°$ gives θ as roughly $109.5°$.

4. Stewart 12.4.6

[2 pts]

Find the cross product of $\boldsymbol{a} = \boldsymbol{i} + e^t\boldsymbol{j} + e^{-t}\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + e^t\boldsymbol{j} - e^{-t}\boldsymbol{k}$ and verify that it is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Solution:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & e^t & e^{-t} \\ 2 & e^t & -e^{-t} \end{vmatrix} = \langle -2, -(-3e^{-t}), -e^t\rangle$$

You can check this is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ by taking its dot product with each of them and making sure you get zero.

5. Stewart 12.5.30

[3 pts]
Find the equation of the plane that contains the the line $x = 3 + 2t, y = t, z = 8 - t$ and is parallel to the plane $2x + 4y + 8z = 17$.

Solution: Finding the equation of a plane requires two things: a point on the plane and a normal vector to the plane. We can find a point on the plane by choosing any point on the given line, since the whole lines lies in the plane. Tkaing $t = 0$ gives the point $P = \langle 3, 0, 8\rangle$. Since our plane is parallel to the given plane, it has the same normal vectors. One normal vector can be found by just reading off the coefficients of x, y, z in the equation of the plane. Thus $\mathbf{n} = \langle 2, 4, 8\rangle$ is a normal vector to the given plane, and hence to our plane, also. To find the equation of the desired plane, we set

$$\mathbf{n} \cdot (\langle x, y, z\rangle - P) = 0,$$

which gives $2x + 4y + 8z = 70$, or $x + 2y + 4z = 35$.

6. Stewart 13.2.18

[2 pts] *Find the unit tangent vector to the curve $\boldsymbol{r}(t) = \langle 4\sqrt{t}, t^2, t\rangle$ at $t = 1$.*

Solution:

The tangent vector at $t = 1$ is found by computing $\mathbf{r}'(t) = \langle \frac{2}{\sqrt{t}}, 2t, 1\rangle$ and evaluating at $t = 1$: $\mathbf{r}'(1) = \langle 2, 2, 1\rangle$. To get the UNIT tangent vector, divide $\mathbf{r}'(1)$ by its length, $|\mathbf{r}'(1)| = \sqrt{2^2 + 2^2 + 1} = 3$. Thus the unit tangent vector at $t = 1$ is $\langle 2/3, 2/3, 1/3\rangle$.

7. Stewart 13.2.50

[2 pts]
If a curve has the property that the position vector $\boldsymbol{r}(t)$ is always perpendicular to the tangent vector $\boldsymbol{r}'(t)$, show that the curve lies on a sphere with center the origin.

Solution: The converse to this statement is proved in Stewart as example 4 in 13.2. One solution is to adapt that proof in the reverse direction. Here is another, slightly more concrete version.

Begin by writing $\mathbf{r}(t)$ in terms of its component functions: $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. What we need to show is that $\sqrt{x(t)^2 + y(t)^2 + z(t)^2} = r^2$ for some r: this says exactly that for all t the point $\mathbf{r}(t)$ lies on a sphere of radius r. We are given that $\mathbf{r}$ is orthogonal to $\mathbf{r}'$, so

$$\langle x(t), y(t), z(t)\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle = 0$$

Expand out the dot product and multiply this equation by two to give the equation

$$2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$$

Now, if we integrate both sides of this equation with respect to t we get

$$x(t)^2 + y(t)^2 + z(t)^2 = C,$$

where C is a constant of integration. Since the left-hand side is non-negative, so is C. Therefore this equation is exactly what we wanted to show, by putting $r = \sqrt{C}$.