ASSIGNMENT 1 SOLUTION

JAMES MCIVOR

1. Supplementary Problems 1, # 4.1

(3 points) Suppose I asked you to draw the figure in $\mathbb{R}^2$ cut out by the two equations $x^2 + y^2 = 1$ and $y + x = 3$. This would be a trick question: there’s nothing to draw! Give an algebraic explanation (i.e., one having to do with solving equations) and a geometric explanation for why there’s nothing to draw.

**Algebraic Reason:** The figure cut out by the two equations consists of all the points $(x, y)$ which satisfy both equations at once. To solve both equations simultaneously, we can write the second equation as $y = 3 - x$, and substitute into the first, giving $x^2 + (3 - x)^2 = 1$. This is equivalent to $2x^2 - 6x + 8 = 0$, or $x^2 - 3x + 4 = 0$. This is a quadratic equation; since the discriminant $b^2 - 4ac = 9 - 16 = -7$ is less than zero, there are no real solutions. Therefore our figure, which is a picture of the solutions, is empty!

**Geometric Reason** The figure we want is the intersection of the line cut out by $x + y = 3$ and the circle cut out by $x^2 + y^2 = 1$. If you draw these, you’ll see that they do not intersect at all!

2. Stewart 10.1.16

(4 points) For the parametrized curve given by $x(t) = \ln t$, $y(t) = \sqrt{t}$ $t \geq 1$:
(a) Eliminate the parameter to find a Cartesian equation of the curve (i.e., the equation that cuts it out).
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increased

**Solution:** (a) Solving both equations for $t$ gives $e^x = t$ and $y^2 = t$, so the equation which cuts out the curve is $e^x = y^2$.
(b) Note that when we just write $\sqrt{t}$, it is assumed that we refer just to the positive square root. Thus for this parametrized curve, the $y$-values are all positive, and we can rewrite the equation that cuts it out as $y = e^{x/2}$. This is just the graph of a function, it looks basically the same as $e^x$. Here’s a picture (the arrow should point up and to the right, i.e. the curve is traced out in the direction of increasing $x$):

3. Stewart 10.1.42

(4 points) If $a$ and $b$ are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point $P$ in the figure, using the angle $\theta$ as the parameter. The line segment $AB$ is tangent to the larger circle.
**Solution** All you have to do is describe the location of $P$ (i.e., give its $x$- and $y$-coordinates) in terms of $a$ and $b$ (which are fixed) and $\theta$ (which varies). First let’s find the $y$-coordinate of $P$: $P$ sits at the same height as the point on the inner circle at angle $\theta$. So its $y$-component is just $b \sin \theta$. The $x$-coordinate of $P$ is the length of the segment $OB$. For this we need to inspect the triangle $OAB$. We are told that $AB$ is tangent to the large circle. This means it forms a right angle with $OA$, since tangents to circles are perpendicular to the radius. So the triangle $OAB$ is a right triangle. We know one of its side lengths, that’s just $a$, and we want the length of the hypotenuse, which we’ll call $x$. Since $\cos \theta = a/x$, we have $x = a \sec \theta$. Thus the parametrization is $(a \sec \theta, b \sin \theta)$.

This is intuitively plausible, since from the picture you can imagine that as $\theta \to \pi/2$, the point $P$ moves off to infinity in the $x$-direction. This is accounted for by the singularity of $\sec \theta$ at $\pi/2$.

4. Stewart 10.2.20

(4 points) Find the points on the curve $x = \cos 3\theta$, $y = 2 \sin \theta$ where the tangent is horizontal or vertical. Graph the curve.

**Solution:** Firstly let’s just make some qualitative observations. Since sin and cos always lie between -1 and 1, the $x$-components are going to oscillate between -1 and 1, while the $y$-components will oscillate between -2 and 2. Moreover, the presence of the 3 in the argument of cos means that the $x$-component will oscillate more frequently than the $y$-component. Also, since sin and cos are periodic, the curve will be a closed loop - the parametrization repeats itself every $2\pi$.

That tells us to expect some vertical and horizontal tangents, since a closed loop must be horizontal and vertical at some points. To find them, let’s first set $\frac{dy}{dx} = 0$ to locate the horizontal tangents. Since $\frac{dy}{dt} = 2 \cos \theta$ and $\frac{dx}{dt} = -3 \sin 3\theta$,

$$\frac{dy}{dx} = -\frac{2 \cos \theta}{3 \sin 3\theta} = 0$$

only when $\cos \theta = 0$, which happens at $\theta = \frac{(2n+1)\pi}{2} = \ldots -\pi/2, \pi/2, 3\pi/2, \ldots$ (n represents an arbitrary integer). We only really need to consider $\pi/2$ and $3\pi/2$, since we have observed that the curve repeats itself after $2\pi$. Since the denominator is nonzero at these points, these are exactly the values of $\theta$ where there is a horizontal tangent. The correspond to the points $(0, \pm 2)$ on the curve.

For the vertical tangents, we solve

$$\frac{dx}{dt} = -3 \sin 3\theta = 0$$

giving $3\theta = n\pi$, where $n$ could be any integer. Thus $\theta$ could be any multiple of $\pi/3$. We only need the ones that lie between 0 and $2\pi$, namely $\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$. What points on our curve do these correspond to? In the same order, they are: $(1,0)$, $(-1, \sqrt{3})$, $(1, \sqrt{3})$, $(-1,0)$, $(1, -\sqrt{3})$, and $(-1, -\sqrt{3})$. Here’s what the curve looks like: