ASSIGNMENT 11 SOLUTION

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1. Stewart 16.4.21

[5 pts]

(1) *If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that*

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1$$

(2) *If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), \dots, (x_n, y_n)$, show that the area of the polygon is*

$$A = \sum_{i=1}^{n-1} \frac{1}{2}(x_iy_{i+1} - x_{i+1}y_i) + \frac{1}{2}(x_ny_1 - x_1y_n)$$

Solution:

(1) The segment is parametrized by $\mathbf{r}(t) = \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)\rangle$ $(0 \le t \le 1)$, so $\mathrm{d}y = (y_2 - y_1)\,\mathrm{d}t$ and $\mathrm{d}x = (x_2 - x_1)\,\mathrm{d}t$. Thus

$$\begin{aligned}\int_C x\,\mathrm{d}y - y\,\mathrm{d}x &= \int_0^1 (x_1 + t(x_2 - x_1))(y_2 - y_1)\,\mathrm{d}t - \int_0^1 y_1 + t(y_2 - y_1)(x_2 - x_1)\,\mathrm{d}t \\ &= (y_2 - y_1)(x_1 + \frac{1}{2}(x_2 - x_1)) - (x_2 - x_1)(y_1 + \frac{1}{2}(y_2 - y_1)) \\ &= \frac{1}{2}[x_1y_2 - x_1y_1 + x_2y_2 - x_2y_1] - \frac{1}{2}[x_2y_1 - x_1y_1 + x_2y_2 - x_1y_2] \\ &= x_1y_2 - x_2y_1\end{aligned}$$

(2) Let C be the boundary of this polygon, so C consists of n segments C_k, each joining (x_k, y_k) to (x_{k+1}, y_{k+1}), except for C_n, which joins (x_n, y_n) to (x_1, y_1). According to the area folrmula derived from Green's theorem, the area of the polygon is

$$A = \frac{1}{2}\int_C y\,\mathrm{d}x - x\,\mathrm{d}y = \sum_{k=1}^{n} \frac{1}{2}\int_{C_k} y\,\mathrm{d}x - x\,\mathrm{d}y$$

By (a), for each $1 \le k \le n-1$ we have $\int_{C_k} x\,\mathrm{d}y - y\,\mathrm{d}x = x_ky_{k+1} - x_{k+1}y_k$, and also $\int_{C_n} x\,\mathrm{d}y - y\,\mathrm{d}x = x_ny_1 - x_1y_n$. Adding these up gives the desired result.

2. Stewart 16.5.37

[5 pts] *This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\boldsymbol{w} = \omega\boldsymbol{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\boldsymbol{r} = \langle x, y, z\rangle$ be the position vector of P.*

(a) *By considering the angle θ in the figure, show that the velocity field of B is given by $\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{r}$.*

(b) Show that $\boldsymbol{v} = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}$.

(c) Show that curl $\boldsymbol{v} = 2\boldsymbol{w}$.

Solution:

(a) To show that $\mathbf{v} = \mathbf{w} \times \mathbf{r}$, we show these vectors have the same direction and smae length. The velocity vector at each point is perpendicular to the line o the z-axis, and so does $\mathbf{w} \times \mathbf{r}$, by the right hand rule. The length of $\mathbf{v}$ is the tangential speed, which is ωd. On the other hand, the length of $\mathbf{w} \times \mathbf{r}$ is $|\mathbf{w}||\mathbf{r}|\sin\theta$, since $\mathbf{w}$ is in the direction of the positive z-axis. But $|\mathbf{r}|\sin\theta = d$ and $|\mathbf{w}| = \omega$, so $|\mathbf{w} \times \mathbf{r}| = \omega d = |\mathbf{v}|$, so these vectors have the same length and direction, hence are equal.

(b) By putting $\mathbf{w} = \omega\mathbf{k}$ and $\mathbf{r} = \langle x, y, z\rangle$ into the formula for the cross product $\mathbf{w} \times \mathbf{r}$, we get $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$.
(c) Again, a simple calculation using the determinant formula for the curl of $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$ gives $\operatorname{curl}\mathbf{v} = 2\mathbf{w}$.

3. Stewart 16.6.24

[5 pts] *Find a parametric repreentation of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$.*

Solution:

The sphere is described in spherical coordinates by the equation $\rho = 4$, which means we can use the other two spherical coordinates ϕ and θ as parameters, giving

$$\mathbf{r}(\phi, \theta) = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi\rangle$$

We need to impose the constraint that $-2 \leq z \leq 2$. Since $z = 4\cos\phi$, this becomes $-1/2 \leq \cos\phi \leq 1/2$, so ϕ ranges from $\pi/3$ to $2\pi/3$ (since ϕ is always between 0 and π). So the bounds on the parameters are $0 \leq \theta 2\pi$ and $\pi/3 \leq \phi \leq 2\pi/3$.