ASSIGNMENT 10 SOLUTION

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1. Stewart 16.3.18

[4 pts] Find a potential function for \( \mathbf{F}(x, y, z) = \langle e^y, xe^y, (z+1)e^z \rangle \) and use it to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{r}(t) = \langle t, t^2, t^3 \rangle \).

**Solution:** The potential function \( f \) must have \( f_x = e^y \), so integrating with respect to \( x \) gives

\[
 f(x, y, z) = xe^y + g(y, z)
\]

We also must have \( f_y = xe^y + \frac{\partial g}{\partial y} = xe^y \), showing that \( \frac{\partial g}{\partial y} = 0 \), hence \( g \) is a constant with respect to \( y \), i.e., just a function of \( z \) only, call it \( h(z) \). Finally, we must have \( f_z = h'(z) = (z+1)e^z \), and integrating gives \( h(z) = ze^z + C \). Putting this back into the above expression for \( f \) yields

\[
 f(x, y, z) = xe^y + ze^z + C
\]

Then since this field is conservative, we can evaluate the line integral by just computing the values of \( f \) at the endpoints \((1, 1, 1)\) and \((0, 0, 0)\) and subtracting (by the FTC for line integrals). So

\[
 \int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 1, 1) - f(0, 0, 0) = 2e
\]

2. Stewart 16.3.24

[3 pts - please be lenient when grading this one - it is difficult to explain why you think the work is zero along a given closed loop] Is the following vector field conservative? Explain.

**Solution:** Yes. We consider various closed loops, and estimate the work done by the field around these loops. First consider a circle centered at the origin. The positive work done in the first and third quadrants appears to exactly cancel the negative work done in the second and fourth quadrants, so we get zero work around circles at the origin.

Now consider a circle lying completely in the first quadrant. On the right side of the circle, the field does positive work, and on the left side, negative. The field on the right side of the circle is stronger, but pointed nearly directly vertical. Towards the left side of the circle, the field strength is weaker, but the field vectors are nearly opposite the direction of motion, so their (negative) contribution to the work seems as though it may cancel out the work done by the larger field lines on the right. A similar analysis applies to circles lying completely in any of the other three quadrants.

Finally consider a rectangle, for example \([-3, 3] \times [1, 3]\), which sits in the first and second quadrants. Give it the usual counterclockwise orientation. The work done on the left hand edge is basically zero, since the field is almost perpendicular to that edge. Along the right hand edge we have a large positive work done by the field, which appears to cancel exactly with the negative work done along the portion of the top edge in the second quadrant. Meanwhile, the two horizontal
components which lie in the first quadrant both give negative work, whilst the bottom edge in the second quadrant gives a slightly larger positive work, which seems to cancel those two negative contributions.

Since all these estimations seem near to zero, it is reasonable to suppose that the work done around any closed loop is zero, and that therefore this field is conservative. In fact, notice that the field lines are horizontal along the line \( y + x = 0 \). This suggests that the y-component of the field may be some scalar multiple of \( y + x \). Similarly, it’s vertical along the line \( y = x \), so the x-component is probably a multiple of \( y - x \). Indeed if you plot the field \( \langle y - x, y + x \rangle \), you get the picture above.

And this field is in fact conservative, as you can check using the “computational” criterion of this section.

3. Stewart 16.3.32

[3 pts] Determine whether the set \( \{ (x, y) \mid x^2 + y^2 \leq 1 \text{ or } 4 \leq x^2 + y^2 \leq 9 \} \) is (a) open, (b) connected, and (c) simply connected.

Solution: The set is not open since it has a boundary. The boundary of this region consists of three circles about the origin, of radii 1, 2, and 3. The set is not connected, either, since if you take a point in the inner disk, and another point in the outer ring, there is no path between them which does not leave the set. Since it is not connected, it is not simply connected, by our definition of simply connected. Intuitively, you might regard the missing region \( 1 < x^2 + y^2 < 2 \) as a “ring-shaped hole”.

4. Stewart 16.3.33

[5 pts] Let \( F(x, y) = \frac{-yi + xj}{x^2 + y^2} \).

(a) Show that \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).

(b) Show that \( \int_C F \cdot dr \) is not independent of path. Does this contradict Theorem 6?

Solution:

(a) \( \frac{\partial P}{\partial y} = \frac{\frac{\partial}{\partial y}(-x^2 - y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \).

(b) To see that the integral is not path-independent, let \( C_1 \) be the semicircle of radius one centered at the origin in the upper half-plane, beginning at \( (1, 0) \) and ending at \( (-1, 0) \). Let \( C_2 \) be the semicircle of radius one centered at the origin in the lower half-plane, beginning at \( (1, 0) \) and ending at \( (-1, 0) \).

\( C_1 \) is parametrized by \( \langle \cos t, \sin t \rangle, \ 0 \leq t \leq \pi \). Along this curve, \( F = \langle -\sin t, \cos t \rangle \). Using this we calculate

\[
\int_{C_1} F \cdot dr = \int_0^\pi \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \ dt = \int_0^\pi 1 \ dt = \pi
\]

On the other hand, \( C_2 \) is parametrized by \( \langle \cos t, -\sin t \rangle \), along which curve the force field \( F = \langle \sin t, \cos t \rangle \), and the tangent vector is \( \langle -\sin t, -\cos t \rangle \). So we have

\[
\int_{C_2} F \cdot dr = \int_0^\pi \langle \sin t, \cos t \rangle \cdot \langle -\sin t, -\cos t \rangle \ dt = \int_0^\pi -1 \ dt = -\pi
\]

These two integrals do not agree, but this does not violate Theorem 6, since the domain of \( F \) is \( \mathbb{R}^2 \) minus the origin, which is not simply connected - it has a “hole”, namely the origin.