SOLUTION

MATH 53, SUMMER 2011 FINAL EXAM
INSTRUCTOR: JAMES MCIVOR
DATE: AUGUST 12, 2011

Write your name at the top. You have 110 minutes to complete the exam. There are 300 points total - 60 points per question. No calculators, cellphones, earphones, or cheat sheets. There is scratch paper at the back of the exam. If your answer comes out messy, you probably made a mistake - go back and check your work. You may find the results below helpful (you are assumed to know what the regions C, D, and V, and their boundaries, stand for in each case).

(Fundamental Theorem of Calculus for Line Integrals) \( \int_C \nabla f \cdot \, \text{d}r = f(r(b)) - f(r(a)) \)

(Green's Theorem) \( \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \text{d}A = \oint_{\partial D} P \text{d}x + Q \text{d}y \)

(Stokes' Theorem) \( \iint_S \text{curl}\, \mathbf{F} \cdot \mathbf{n} \, \text{d}S = \oint_{\partial S} \mathbf{F} \cdot \text{d}\mathbf{r} \)

(Divergence Theorem) \( \iiint_V \text{Div}\, \mathbf{F} \, \text{d}V = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} \, \text{d}S \)

(Volumes of some familiar solids)

\[ V(\text{tetrahedron}) = \frac{1}{3} A_{\text{base}} h \quad V(\text{sphere}) = \frac{4}{3} \pi r^3 \quad V(\text{cylinder}) = \pi r^2 h. \]
(1) Use Green's theorem to compute

$$\int_C \sqrt{1 + x^2} \, dx + 2xy \, dy,$$

where $C$ is the triangle with vertices $(0,0), (1,0)$, and $(1,3)$, given the counter-clockwise orientation.

\[ \rho = -\frac{x^3}{1 + x^2} \quad \alpha = 2xy \]

\[ \frac{\partial \alpha}{\partial x} - \frac{\partial \rho}{\partial y} = 2y \]

$C = \partial D$, where $D$ is the triangular region \( \{ 0 \leq x \leq 3 \}
\{ 0 \leq y \leq 3x \}. \]

By Green's, the integral is equal to

\[ \iint_D \left( \frac{\partial \alpha}{\partial x} - \frac{\partial \rho}{\partial y} \right) \, dx \, dy = \int_0^1 \int_0^{3x} 2y \, dy \, dx = \int_0^1 (3x)^2 \, dx = 3 \]
Evaluate \( \iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS \), where \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} + yx \mathbf{j} + \cos(ze^{xy}) \mathbf{k} \) and \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 2 \) that lies above the plane \( z = 1 \), oriented outward.

By Stokes' Theorem,
\[
\iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r},
\]
where \( dS \) is the circle of radius 1 on the plane \( z = 1 \), oriented counterclockwise to match the orientation of \( S \).

As \( \mathbf{n} \) is parametrized by \( \mathbf{r}(t) = \langle \cos t, \sin t, 1 \rangle \), we have
\[
\mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle.
\]

Therefore,
\[
\mathbf{F} \cdot \mathbf{r}'(t) = \langle \cos^2 t, \sin t \cos t, \cos z e^{xy} \rangle \cdot \langle -\sin t, \cos t, 0 \rangle
= -\sin t \cos^2 t + \sin t \cos t \cdot 0 = 0 \, dt.
\]

So,
\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int 0 \, dt = \boxed{0}.
\]
(3) (a) Let \( C \) be a level set of the differentiable function \( f(x, y) \). Suppose \( C \) is parametrized by \( r(t) \), for \( 0 \leq t \leq 10 \). Now let \( C' \) be the curve which is also given by the parametrization \( r(t) \), but with \( 2 \leq t \leq 5 \). In other words, \( C' \) is portion of the level set \( C \), but not necessarily all of it. What is \( \int_{C'} \nabla f \cdot dr \)? Justify your answer.

(b) The vector field plotted below is not conservative. By drawing one or more curves in the field, explain why not.

(c) Let \( T(x, y) \) denote the temperature at the point \((x, y)\) on a hot metal plate. A bug is on the hot metal plate, and she knows the partials \( \frac{\partial T}{\partial x} \) and \( \frac{\partial T}{\partial y} \). What should she do to determine a direction in which to walk so that the temperature does not change as she begins walking? Will this always work, no matter what the values of \( \frac{\partial T}{\partial x} \) and \( \frac{\partial T}{\partial y} \) are?

(a) By the FTC for line integrals, \( \int_{C'} \nabla f \cdot dr = f(r(15)) - f(r(2)) \).

Since \( C' \) lies entirely on the level set \( C \) for \( S \),

\( f(r(s)) = f(r(t)) \) (\( = f(r(t)) \) for any \( 0 \leq t \leq 10 \)). Thus

the line integral is zero.

Geometrically, \( \nabla f \) is always \( \perp \) to \( dr \), so the integrand is zero.

(b) The path shown has 4 parts. Along \( C_1 \) and \( C_4 \), the field is \( \perp \) to the curve, so no work is done.

Along \( C_2 \) and \( C_3 \), the field is in the same direction as the curve, so positive work is done. Thus the net work around the loop is positive, so the field is not conservative.

(c) She should compute \( \Delta T \) and move in a direction perpendicular to \( \Delta T \) (along a level set). This won't work if the partials are both \( \geq 10 \).
(4) Find \( \int_{S} F \cdot \mathbf{n} \, dS \), where \( S \) is the boundary of the region enclosed by the plane \( z = 0 \), the cylinder \( x^2 + y^2 = 1 \), and the hemisphere \( x^2 + y^2 + (z - 1)^2 = 1 \) \((z \geq 1)\), and \( F(x,y,z) = (x, xz^2, y^2 \sin x) \).

\[ S = \partial E, \text{ where } E \text{ is a solid cylinder of radius one and height one, with a hemisphere of radius one on top.} \]

By Div Thm,

\[ \int_{S} \nabla \cdot F \, dS = \iiint_{E} \text{Div} \, F \, dV \]

\[ \text{Div} \, \frac{\mathbf{x}}{r} = 1, \quad r = \sqrt{x^2 + y^2 + z^2} \]

\[ \int_{S} \frac{\mathbf{x}}{r} \cdot \mathbf{n} \, dS = \iiint_{E} dV = \text{Vol} \,(E) \]

\[ = \text{Vol} \,(\text{cylinder}) + \text{Vol} \,(\text{hemisphere}) \]

\[ = (\pi (1)^2) \cdot 1 + \frac{1}{2} \cdot \frac{4}{3} \pi (1)^3 \]

\[ = \pi + \frac{2}{3} \pi = \frac{5}{3} \pi \]
Use Stokes' Theorem to compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F}(x, y, z) = (xz, xy, yz) \) and \( C \) is the piecewise linear path which goes from \((1,0,0)\) to \((0,1,0)\) to \((0,0,1)\) and back to \((1,0,0)\) again.

\( C \) is the shaded triangular region. By the right-hand rule, the orientation of \( S \) must be upward to match that of \( C \).

By Stokes,
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S}
\]

\[
\text{curl} \mathbf{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x^2 & xy & yz
\end{vmatrix} = \langle z, x, y \rangle
\]

The equation of \( S \) is \( x + y + z = 1 \), so we can parametrize \( S \) by \( \mathbf{r}(x, y) = \langle x, y, 1-x-y \rangle \), with \( \mathbf{r}_x \times \mathbf{r}_y = \langle 1, 1, 1 \rangle \).

The parameters \( x, y \) range over the triangle \( T \) in the \( xy \)-plane.

\[
\text{curl} \mathbf{F} \cdot d\mathbf{S} = \langle 1-x-y, x, y \rangle \cdot \langle 1, 1, 1 \rangle \, dx \, dy
\]

\[
= \int_0^1 \int_0^{1-x} \, dx \, dy
\]

So \( \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_T \, dx \, dy = A(T) = \frac{1}{2} \).