MATH 115, SUMMER 2012
QUIZ 5 SOLUTION

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There are problems on both sides of this page, 20 points total. Write clearly and in complete sentences. If you need extra paper, ask me.

(1) (2 points each) True or False. Explain your answer.

(a) There are infinitely many Pythagorean triples containing the integer 12.

False: It's enough to show that there are only finitely many primitive triples containing 12. Consider a primitive Pythagorean triple x, y, z, one of which is 12. We saw in class that x and z must be odd, so $y = 12$, furthermore, we know there exist integers r and s such that $2rs = y = 12$, and there are only finitely many values of r, s that satisfy this.

(b) If a line L in $\mathbb{R}^2$ whose slope is a rational number passes through a point (a, b), and a and b are not rational numbers, then L has no rational points.

False: Conside the line L through the origin with slope 1. It passes through $(\sqrt{2}, \sqrt{2})$, whose coordinates are not rational, but it still has other rational points, e.g., $(1, 1)$.

(c) If P is a property of a positive integer which is such that, whenever n has property P, then $n - 2$ has property P, then there are no positive integers with property P.

True: This is an example of the method of descent. The question was a little unclear as stated. I should have said that having property P entails being a positive integer. Consequently I was lenient with grading, as long as you explained your answer.

(d) If an integer matrix A has nonzero determinant, then there is another integer matrix B such that $AB = BA = I$.

False: To be an invertible integer matrix, the determinant must be ± 1, not just nonzero.

(e) If C is a curve defined by a degree 2 polynomial $f(x, y)$, P is a rational point on C, and L is a line through P whose slope is a rational number, then L intersects C in another point, which is also a rational point on C.

False: Let C be the parabola $x = y^2$, $P = (0, 0)$, and L be the line $y = 0$. Then L meets C in exactly one rational point. Alternatively, it could be that the line in question is tangent to C.

(2) (5 points) Let a and b be relatively prime integers. Explain how to choose c and d such that the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is in the modular group.

Solution: Since a and b are coprime, we can find integers x and y such that $ax + by = 1$. Now the determinant of the given matrix is $ad - bc$, so if we take $d = x$ and $c = -y$, this determinant will be 1, and the matrix will be in the modular group.

(3) (5 points) Explain why the equation $7x^3 + 8y^2 = 818$ has no integer solutions.

Solution: Suppose there *were* integers x, y satisfying this equation. First observe that x must be even, say $x = 2k$, so our equation can be written as

$$7 \cdot 8k^3 + 8y^2 = 818.$$

Look at the equation mod 8: you get $0 \equiv 818 \equiv 2 \mod 8$, which is a contradiction. So there can be no solution in integers.