MATH 115, SUMMER 2012
QUIZ 4 SOLUTION

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There are two problems - one is on the back of this page, 20 points total. Write clearly and in complete sentences. If you need extra paper, ask me. You may find the following definition and theorems useful:

Definition 1. A positive definite form $f(x, y) = ax^2 + bay + cy^2$ is **reduced** if either $-a < b \leq a < c$ or $0 \leq b \leq a = c$.

Theorem 1 (The Reduction Theorem)**.** *Let $f(x, y) = ax^2+bxy+cy^2$ be a primitive positive definite QF (integral and binary as usual). Then*

(1) *f is equivalent to a unique reduced form.*
(2) $|b| \leq a \leq \sqrt{-d/3}$

Theorem 2. *Let p be and odd prime and d any integer congruent to 0 or 1 mod 4. Then p is represented by a form of discriminant d if and only if d is a square mod p.*

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(1) (10 points) Find a reduced form equivalent to $37x^2 - 22xy + 13y^2$.

Solution: $13x^2 - 4xy + 28y^2$, found by appling first step one (using S), and then step 2 (using T^{-1}).

(2) (10 points) Which primes are represented by the form $f(x, y) = x^2 + 3y^2$? Use Theorems 1 and 2 to carefully prove your answer.

Solution: First of all observe that $p = 2$ is definitely not represented, and $p = 3$ *is*, so from now on assume $p > 3$. Now, f has discriminant -12; we begin by determining all primitive reduced forms of this discriminant. So we look for values of a, b, c that satisfy the two conditions

$$12 = 4ac - b^2, \qquad |b| \leq a \leq \sqrt{12/3} = 2,$$

where the second condition comes from part 2 of Thm 1. We consider the possible values of $|b|$. It can't be 1, since then we would have $13 = 4ac$, which is impossible. If $|b| = 2$, then the second condition forces $a = 2$, and solving for c in the first equation gives $c = 2$. This gives us the form $2x^2 + 2xy + 2y^2$, which is reduced, but is not primitive. It can't represent any primes but 2, since its coefficients are all divisible by 2.

Finally, if $|b| = 0$, then $12 = 4ac$, so one of a or c must be three and the other is one. Reducedness forces $a = 1, c = 3$, giving the form f.

We know $p = 2$ is not represented by f, so let p be an odd prime. Thm 2 says that a prime p is represented by f if and only if -12 is a square mod p, so we calculate

$$\left(\frac{-12}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^2\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}\left(\frac{p}{3}\right)(-1)^{(p-1)/2} = \left(\frac{p}{3}\right)$$

So this says that -12 is a square mod p if and only if $p \equiv 1 \mod 3$. Note that we needed the fact that $p \neq 3$ in applying quadratic reciprocity.

Thus p is represented by f if and only if $p = 3$ or $p \equiv 1 \mod 3$.