MATH 115, SUMMER 2012
LECTURE 26

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1. The Projective Plane

We now construct a two-dimensional projective space - it's just like before, but with one extra variable.

Definition 1. The **projective plane over** $\mathbb{R}$, denoted $\mathbb{P}^2(\mathbb{R})$, is the set of lines through the origin in $\mathbb{R}^3$.

- As before, points in $\mathbb{P}^2$ can be described in homogeneous coordinates, but now there are three (nonzero!) coordinates.
- We write $[x : y : z]$ for a point in $\mathbb{P}^2$. It corresponds to the line through the vector (x, y, z) in $\mathbb{R}^3$.
- But there are many ways to write the same point: $[x : y : z] = [\lambda x : \lambda y : \lambda z]$ where λ is any nonzero real number.
- draw a sphere in $\mathbb{R}^3$. Each line through the origin meets the sphere twice. Choose one point out of the two for each line, say the one in the top half of $\mathbb{R}^3$. For points that lie in the xy-plane, choose the point with $x > 0$. For the two points on the y-axis, choose the point $(0, 1, 0)$.
- thus we see that the projective plane looks like a hemisphere.
- but like before, if you walk to the edge of the hemisphere, you reappear on the polar opposite side!
- we can break $\mathbb{P}^2$ into some simpler pieces. consider the following three subsets (which overlap):

$$U_0 = \{[x : y : z] \mid x \neq 0\}, \quad U_1 = \{[x : y : z] \mid y \neq 0\}, \quad U_2 = \{[x : y : z] \mid z \neq 0\},$$

- Look for example at U_0. If we have a point $[x : y : z]$ with $x \neq 0$, we may divide all three homogeneous coordinates by x to obtain the equivalent representation of the same point $[1 : \frac{y}{x} : \frac{z}{x}]$.
- writing $u = \frac{y}{x}$ and $v = \frac{z}{x}$, we have the point $[1 : u : v]$. Here as y and z vary, u and v range over all real numbers.
- So we see that U_0 is in bijection with $\mathbb{R}^2$. So also are U_1 and U_2.
- Thus $\mathbb{P}^2$ is the union of three copies of $\mathbb{R}^2$, all of which overlap.

2. Curves in $\mathbb{P}^2(\mathbb{R})$

Just like above, it doesn't make sense to plug in homogeneous coordinates to a function, since if we scale the coordinates, the point doesn't change, but the output of the function does!

However, by arguing as we did above for $\mathbb{P}^1$, if $F(x, y, z)$ is a homogeneous polynomial, the set of points $[x : y : z] \in \mathbb{P}^2$ such that $F(x, y, z) = 0$ does make sense. This is what we call a curve.

Definition 2. A **projective plane curve** of degree d is a set of points of the form

$$C_F = \{[x : y : z] \mid F(x, y, z) = 0\}$$

where F is a homogeneous polynomial of degree d.

A curve of degree one is called a line. A curve of degree two a conic, etc....

Example 2.1. Let $F(x, y, z) = x^2 + y^2 - z^2$. This is a homogeneous polynomial of degree 2. To see how the corresponding curve C looks in $\mathbb{P}^2$, we study how it looks in each of the three subsets U_0, U_1, U_2, and then piece these pictures together.

Let's do U_2 first. In U_2, we have $z = 1$, so our equation is just $x^2 + y^2 = 1$. So U_2 is just a copy of $\mathbb{R}^2$, and there the curve C is just a circle.

In U_0, we have $x = 1$, so our equation becomes $z^2 - y^2 = 1$, so C looks like a hyperbola here!

In U_1, we have $y = 1$, so our equation becomes $z^2 - x^2 = 1$, so C looks like a hyperbola here, too.

In U_2, we have $z = 1$, so our equation becomes $x^2 + y^2 = 1$, so C looks like a circle here.

- So we see one quadratic equation, which looks like different conic sections depending on where in $\mathbb{P}^2$ we look.

- This settles the motivational question addressed at the beginning of yesterday's lecture - why can quadratic equations generate shapes (parabolas, ellipses, hyperbolas) which look so different but all come from the same type of equation?

- the answer is simply that they only look different in $\mathbb{R}^2$ - if we expand our geometry to include the points at infinity, then the shapes are all the same.[1]

Try to draw this on the hemisphere

- explain how to visualize U_0, U_1, U_2 by projecting the sphere onto the planes $x = 1$, $y = 1$, $z = 1$.

Here is one thing that made the discovery of projective space so surprising - in fact it caused a huge philosophical controversy!

Example 2.2. In the projective plane, there are no parallel lines! That is to say, every pair of lines intersect. In fact, if the lines are distinct, then they intersect in a *unique* point!

Exercise: prove this.

[1]There is actually still a problem - the parabola. In $\mathbb{R}^2$, it has only one branch, while the hyperbola has two. The reason, roughly, is that $y = x^2$ doesn't have real solutions when y is negative. However, we can resolve this by working over $\mathbb{C}$. But then things become more difficult to visualize.