- first finish the end of previous lecture, which says roughly that if a line meets a curve of degree \(d\) in more than \(d\) points, then the line is actually a component of the curve.

- Today we study projective space in dimensions one and two.
- Motivation: Consider the standard conics (degree 2 curves): circle, ellipse, hyperbola, parabola.
  - The first two are “closed loops”, but the second two aren’t.
  - This dramatic geometric difference is very strange, since algebraically they’re all just quadratic equations.
  - Idea: we expand our geometry to include some extra “points at infinity”, in such a way that even the hyperbola and parabola “close up”, so that all conics basically look the same.

1. The Projective Line

**Definition 1.** The **projective line over** \(\mathbb{R}\), denoted \(\mathbb{P}^1(\mathbb{R})\), is the set of all lines through the origin in \(\mathbb{R}^2\).

A nonzero point \(P\) in \(\mathbb{R}^2\) determines a point of \(\mathbb{P}^1(\mathbb{R})\) - namely the line through the origin and \(P\).

But different points in \(\mathbb{R}^2\) give us the same point of \(\mathbb{P}^1(\mathbb{R})\),

More precisely, two points in \(\mathbb{R}^2\) determine the same point of \(\mathbb{P}^1\) if they lie on the same line.

So \((a, b)\) and \((c, d)\) determine the same point in \(\mathbb{P}^1\) if \((a, b) = \lambda(c, d) = (\lambda c, \lambda d)\) for some nonzero scalar \(\lambda\).

Thus we arrive at the following equivalent definition of \(\mathbb{P}^1(\mathbb{R})\):

**Alternate Definition** \(\mathbb{P}^1(\mathbb{R})\) is the set of nonzero vectors \((x, y)\), where \(x, y \in \mathbb{R}\), and two vectors \((x, y)\) and \((x', y')\) are considered the same if \(x' = \lambda x\) and \(y' = \lambda y\) for some \(\lambda \neq 0\).

We can visualize \(\mathbb{P}^1\) as follows. For each line through the origin, it meets the unit circle exactly twice.

- for each such line, pick the point on the circle in the right half of the plane (for the vertical line, pick \((0, 1)\)
- thus \(\mathbb{P}^1\) is in bijection with this upper hemisphere.
- but something’s funny - if you walk around the semicircle, and go all the way to the left, you return to the point on the far right!

- Draw picture
2. Homogeneous Coordinates on $\mathbb{P}^1(\mathbb{R})$

By the alt definition above, we can write points in $\mathbb{P}^1$ as nonzero vectors, but the vectors are a little ambiguous, because multiples of the same vector count as the same point. If $(x, y)$ is a nonzero vector in $\mathbb{R}^2$, we write the corresponding point of $\mathbb{P}^1$ as $[x : y]$, and we call $x$ and $y$ the **homogeneous coordinates** on $\mathbb{P}^1$.

- Important: the homogeneous coordinates cannot both be zero, and they are only defined up to scalars.
  - For example, $[3 : 2] = [6 : 4]$ in $\mathbb{P}^1$, and $[0 : 0]$ is not a point of $\mathbb{P}^1$, but $[0 : 1]$ and $[1 : 0]$ are.
  - we distinguish two types of points:
    - if $x \neq 0$, we may divide through both coordinates by $x$ and get $[1 : \frac{y}{x}]$. Every point $[x : y]$ with $x$ nonzero may be written this way.
    - on the other hand, if $x = 0$, then our point looks like $[0 : y] = [0 : 1]$ - we are allowed to divide by $y$ because $y \neq 0$, since both coordinates are not allowed to be zero. So we have:
      $$\mathbb{P}^1(\mathbb{R}) = \{[1 : m] | m = \frac{y}{x} \in \mathbb{R} \} \cup \{0 : 1\}$$
  - The set of points $\{[1 : m]\}$ is in bijection with $\mathbb{R}$, since $m$ can be any real number. So $\mathbb{P}^1$ can be written as the union of $\mathbb{R}$ with one extra point $[0 : 1]$, which we call the “point at infinity”.

3. Functions on $\mathbb{P}^1$

- If we have a polynomial in two variables $x, y$, we can think of it as a function on $\mathbb{R}^2$ - indeed, for any point $(x, y)$, we plug it into $f$, and it outputs a real number.
  - So can we do the same with the homogeneous coordinates on $\mathbb{P}^1$? That is, given $[x : y] \in \mathbb{P}^1(\mathbb{R})$, can we plug it into $f$ and get a number.
  - not really, because the points $[x : y]$ and $[2x : 2y]$ are the same point in $\mathbb{P}^1$, but when we plug in $f(x, y)$ and $f(2x, 2y)$, we’ll probably get two different answers. So this function is not well-defined!
  - If we have a homogeneous polynomial $F$, i.e., where all terms have the same degree $d$, then you can check that $F(2x, 2y) = 2^d F(x, y)$. More generally, $F(\lambda x, \lambda y) = \lambda^d F(x, y)$ for any nonzero scalar $\lambda$. So this is still not well-defined!
  - however, it does make sense to talk about the points where $F$ is zero. For suppose we choose a point $[x : y]$ in $\mathbb{P}^1$, and plug it into the homogeneous $F$ and get $F(x, y) = 0$. Now choose a different way of writing the same point, say $[x : y] = [5x : 5y]$. Then we want to see that $F$ is zero when we plug in this other way of writing the same point. It is, since $F(5x, 5y) = 5^d F(x, y) = 5^d \cdot 0 = 0$.
  - conclusion: polynomials, even homogeneous ones, DO NOT give function on $\mathbb{P}^1$, like they do on $\mathbb{R}^2$.
  - but for homogeneous polynomials $F$, the set of points in $\mathbb{P}^1$ where $F$ is zero does make sense. We will use this to define curves in $\mathbb{P}^2$ in the next section.