MATH 115, SUMMER 2012
LECTURE 26

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- first finish the end of previous lecture, which says roughly that if a line meets a curve of degree d in more than d points, then the line is actually a component of the curve.

-Today we study projective space in dimensions one and two.

-Motivation: Consider the standard conics (degree 2 curves): circle, ellipse, hyperbola, parabola.

-The first two are "closed loops", but the second two aren't.

-This dramatic geometric difference is very strange, since algebraically they're all just quadratic equations.

- Idea: we expand our geometry to include some extra "points at infinity", in such a way that even the hyperbola and parabola "close up", so that all conics basically look the same.

1. The Projective Line

Definition 1. The **projective line over** $\mathbb{R}$, denoted $\mathbb{P}^1(\mathbb{R})$, is the set of all lines through the origin in $\mathbb{R}^2$.

A nonzero point P in $\mathbb{R}^2$ determines a point of $\mathbb{P}^1(\mathbb{R})$ - namely the line through the origin and P.

But different points in $\mathbb{R}^2$ give us the same point of $\mathbb{P}^1(\mathbb{R})$,

More precisely, two points in $\mathbb{R}^2$ determine the same point of $\mathbb{P}^1$ if they lie on the same line.

So (a, b) and (c, d) determine the same point in $\mathbb{P}^1$ if $(a, b) = \lambda(c, d) = (\lambda c, \lambda d)$ for some nonzero scalar λ.

Thus we arrive at the following equivalent definition of $\mathbb{P}^1(\mathbb{R})$:

Alternate Definition $\mathbb{P}^1(\mathbb{R})$ is the set of *nonzero* vectors (x, y), where $x, y \in \mathbb{R}$, and two vectors (x, y) and (x', y') are considered the same if $x' = \lambda x$ and $y' = \lambda y$ for some $\lambda \neq 0$.

We can visualize $\mathbb{P}^1$ as follows. For each line through the origin, it meets the unit circle exactly twice.

- for each such line, pick the point on the circle in the right half of the plane (for the vertical line, pick $(0, 1)$

- thus $\mathbb{P}^1$ is in bijection with this upper hemisphere.

- but something's funny - if you walk around the semicircle, and go all the way to the left, you return to the point on the far right!

- Draw picture

2. Homogeneous Coordinates on $\mathbb{P}^1(\mathbb{R})$

By the alt definition above, we can write points in $\mathbb{P}^1$ as nonzero vectors, but the vectors are a little ambiguous, because multiples of the same vector count as the same point. If (x, y) is a nonzero vector in $\mathbb{R}^2$, we write the corresponding opint of $\mathbb{P}^1$ as $[x : y]$, and we call x and y the **homogeneous coordinates** on $\mathbb{P}^1$.

- Important: the homogeneous coordinates cannot both be zero, and they are only defined up to scalars.

- For example, $[3 : 2] = [6 : 4]$ in $\mathbb{P}^1$, and $[0 : 0]$ is *not* a point of $\mathbb{P}^1$, but $[0 : 1]$ and $[1 : 0]$ are.

- we distinguish two types of points:

- if $x \neq 0$, we may divide through both coordinates by x and get $[1 : \frac{y}{x}]$. Every point $[x : y]$ with x nonzero may be written this way.

- on the other hand, if $x = 0$, then our point looks like $[0 : y] = [0 : 1]$ - we are allowed to divide by y because $y \neq 0$, since both coordinates are not allowed to be zero. So we have:

$$\mathbb{P}^1(\mathbb{R}) = \{[1 : m] \,|\, m = \frac{y}{x} \in \mathbb{R}\} \cup \{[0 : 1]\}$$

-The set of points $\{[1 : m]\}$ is in bijection with $\mathbb{R}$, since m can be any real number. So $\mathbb{P}^1$ can be written as the union of $\mathbb{R}$ with one extra point $[0 : 1]$, which we call the "point at infinity".

3. Functions on $\mathbb{P}^1$

-If we have a polynomial in two variables x, y, we can think of it as a function on $\mathbb{R}^2$ - indeed, for any point (x, y), we plug it into f, and it outputs a real number.

-So can we do the same with the homogeneous coordinates on $\mathbb{P}^1$? That is, given $[x : y] \in \mathbb{P}^1(\mathbb{R})$, can we plug it into f and get a number.

-not really, because the points $[x : y]$ and $[2x : 2y]$ are the same point in $\mathbb{P}^1$, but when we plug in $f(x, y)$ and $f(2x, 2y)$, we'll probably get two different answers. So this function is not well-defined!

- If we have a *homogeneous* polynomial F, i.e., where all terms have the same degree d, then you can check that $F(2x, 2y) = 2^d F(x, y)$. More generally, $F(\lambda x, \lambda y) = \lambda^d F(x, y)$ for any nonzero scalar λ. So this is still not well-defined!

- however, it does make sense to talk about the points where F is zero. For suppose we choose a point $[x : y]$ in $\mathbb{P}^1$, and plug it into the homogeneous F and get $F(x, y) = 0$. Now choose a different way of writing the same point, say $[x : y] = [5x : 5y]$. Then we want to see that F is zero when we plug in this other way of writing the same point. It is, since$F(5x, 5y) = 5^d F(x, y) = 5^d \cdot 0 = 0$.

- conclusion: polynomials, even homogeneous ones, DO NOT give funstion n $\mathbb{P}^1$, like they do on $\mathbb{R}^2$.

- but for homogeneous polynomials F, the set of points in $\mathbb{P}^1$ where F is zero does make sense. We will use this to define curves in $\mathbb{P}^2$ in the next section.