MATH 115, SUMMER 2012
LECTURE 17

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Today we begin studying quadratic forms, which is a vast and interesting subject of its own, but we will only get a brief glimpse of it over the next week or so.

1. Quadratic Forms

- motivation - we figured out weeks ago which integers can be written as sums of squares

- think of that problem this way: let $f(x, y) = x^2 + y^2$. What are the outputs (range) of this function f, as x and y range over all integers?

- question: if we replacve f above by some other quadratic function of two variables, how does the answer change?

- for example, if $f(x, y) = 2x^2 - y^2$, what are the outputs of f?

Definition 1. A **form** (also called a **homogeneous polynomial**) in n variables is a polynomial $f(x_1, \ldots, x_n)$ in which every term has the same degree. In general, the coefficients may be integers, rational numbers, real numbers, etc., but for us, they will usually be integers unless stated otherwise.

A **quadratic form** is a form of degree two. Binary means the form has two variables, ternary means it has three, etc.

Examples 1.1. - $f(x, y) = x^2 + y^2$ is a binary quadratic form, but $f(x, y) = x^2 + y^2 - 1$ is not.

- $f(x, y, z) = x^2 - y^2 - z^2$ is a ternary quadratic form, whereas $f(x, y, z) = x^3 + xyz + xyz^2$ is not a form at all.

To relate this notion to the "sums of squares" problem:

Definition 2. If n is any integer, and $f(x_1, \ldots, x_n)$ is a quadratic form, we say f **represents** n if there are integers $a_1, \ldots, a_n$ such that $f(a_1, \ldots, a_n) = n$. We say f **properly represents** n if the gcd of the a_i is 1.

Note - If f represents n, but not properly, then gcd of the a_i is some $g > 1$, and dividing through by g gives: f represents n/g^2 properly. So it's enough to consider proper representations.

- **main question**: given a form f, which integers does it represent?
- for now focus on binary quadratic forms:
- much of this can be found by looking at the discriminant:

2. Discriminants

Definition 3. Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form. The **discriminant** of f is $d = b^2 - 4ac$.

- the discriminant can tell us when f has nontrivial zeroes:

Theorem 1. *Let f, d be as in above definition. If d is nonzero and not a perfect square, then the only solution of $f(x, y) = 0$ is $x = y = 0$.*

Proof. - note that if a or c is zero, then $d = b^2$, contradicting our assumption that d is not a square. So they're both nonzero.

- If either of x or y is zero, then so is the other.
- assume there is a point $(x, y) \neq (0, 0)$ with $f(x, y) = (0, 0)$. We look for a contradiction.
- key formula:

$$4af(x, y) = (2ax + by)^2 - dy^2$$

- set equal to zero, and get d is a perfect square. Contradiction.

□

3. Definiteness

The first thing we can ask about a form is whether its outputs are always positive, always negative, or both.

Definition 4. The form $f(x, y)$ is **indefinite** if it has both positive and negative outputs. It's **positive semidefinite** is all its outputs are ≥ 0. It's **positive definite** if all its outputs are *strictly* greater than 0. Define negative (semi-)definite similarly.

Good news: the discriminant can tell us whether a form is definite or not.

Theorem 2. *Let f be a binary quadratic form with discriminant d.*

1. *If $d > 0$ the f is indefinite.*
2. *If $d = 0$ then f is semidefinite, but not definite.*
3. *If $d < 0$, then the coefficients a and c have the same sign, and f is positive definite if they're both positive and negative definite if they're both negative.*

Proof. Boring - see book. Deal with three cases separately and be clever with the "key formula" from the last proof.

□

Which integers d can be discriminants of some form? The answer is actually easy:

Proposition 1. *There is a form having discriminant d if and only if d is congruent to 0 or 1 mod 4.*

Proof. - assume there is a form with discriminant d. Then the integer d can be written as $b^2 - 4ac \equiv b^2$ mod 4. But squares mod 4 are always 0 or 1.

- other way: suppose d is congruent to 0, say $d = 4k$. Set $f = x^2 - ky^2$. it has discriminant $4k$
- now suppose $d \equiv 1$ mod 4, say $d = 4k + 1$. Set $f = x^2 + xy - ky^2$ it has disc d.

□

4. Lattices

To introduce some of the ideas of linear algebra, think of forms in the following way:

Let

$$\mathbb{Z}^2 = \{(a,b) \mid a,b \in Z\}$$

- It's a subset of $\mathbb{R}^2$ (draw picture)
- We think of a binary form as a function from $\mathbb{Z}^2 \to \mathbb{Z}$
- Can regard two integers x, y as a vector $\mathbf{x} = (x, y)$. Draw it.
- Consider the expression:

$$\mathbf{x}^T A\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- expand it: get $ax^2 + bxy + cxy + dy^2$. This is a quadratic form.
- other way, given $ax^2 + bxy + cy^2$, can write it using a matrix

$$A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

Notice it's symmetric ($A = A^T$)
- so can interchange the two:

$$\text{binary quadratic forms} \quad \leftrightarrow \quad \text{symmetric } 2 \times 2 \text{ integer matrices}$$

- now check that if d is defined as above, then $d = -4 \det A$.
- use this to prove part (2) of previous theorem on discriminants and definiteness.

——————————— Problem Session ———————————

- For each form, find the discriminant and say whether it's indefinite/pos/neg semidefinite, or pos/neg definite:

(1) $f(x,y) = 2x^2 - xy$
(2) $f(x,y) = x^2 + 3xy + y^2$.
(3) $f(x,y) = x^2 - 2xy + y^2$

- For which of the values $d = -1, 0, 1$ is there a quadratic form having that discriminant? Write down an explicit form in each case.
- If a form factors as $f(x,y) = (\alpha x + \beta y)(\gamma x + \delta y)$, show that it must be semidefinite. Show further that the discriminant of such a form must be a perfect square (maybe 0).
- Prove that a polynomial $f(x,y)$ is homogeneous (of degree d) if and only if $f(\lambda x, \lambda y) = \lambda^d f(x,y)$.