MATH 115
SUMMER 2012
PRACTICE FOR FINAL EXAM

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(1) Prove, without using the Pythagorean triples theorem, that if x, y, z are integers whose gcd is one, satisfying $x^2 + y^2 = z^2$, then z must be odd.

(2) Find all positive integers n such that $10 \mid n^{10} + 1$.

(3) By using the chinese remainder theorem, or by any other method, find all solutions to the congruence

$$x^5 \equiv 5 \mod 12.$$

(4) If $g^4 \equiv -1 \mod 17$, explain why g cannot be a primitive root mod 17.

(5) Describe all pairs of relatively prime integers a, b such that $6ab$ is a perfect square.

(6) Determine whether the following two congruences have solutions:
  (a) $x^2 \equiv 12 \mod 37$
  (b) $x^2 \equiv 80 \mod 33$

(7) Consider the following sequence of quadratic forms:

$$\begin{aligned} f_0(x, y) &= x^2 + y^2 \\ f_1(x, y) &= x^2 - 2xy + 2y^2 \\ f_2(x, y) &= x^2 - 4xy + 5y^2 \\ &\vdots \\ f_k(x, y) &= x^2 - 2kxy + (k^2 + 1)y^2. \end{aligned}$$

  (a) Show that all the f_k represent the same integers.
  (b) Given that $f_0(a, b) = n$, find integers x, y (which possibly depend on any of a, b, n, and k) such that $f_k(x, y) = n$.