MATH 115, SUMMER 2012
WORKSHEET FOR LECTURE 26

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1. Basic Stuff

Mark the following as true or false.

(1) If C is a plane curve, every line in the plane intersects C. **False**
(2) Every curve contains rational points. **False**
(3) A point is a curve. **True**: consider the equation $x^2 + y^2 = 0$, which defines the curve containing only the origin.
(4) The union of three vertical lines is a curve. **True**: say the three lines are $x = -1, 0, 1$, then the equation defining this curve is $x(x-1)(x+1) = 0$.
(5) The equation $5x^2+4y^2 = 2$ has solutions in integers. **False**: look at it mod 4 or 5 to get a contradiction.

2. What do derivatives tell us about curves?

Here we investigate the notion of a **singularity**, which is very important in studying curves.

Draw a picture of the following curves:

(1) $f_1(x, y) = x^2 - y^2 = 0$.
(2) $f_2(x, y) = y^2 - x^3 = 0$.
(3) $f_3(x, y)x^4 - x^2 - y^2 = 0$ (this one is a "figure eight" or "infinity" shape, it may be harder to tell that from the equation)

What do (1) and (3) have in common geometrically? Compute the partial derivatives

$$\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_3}{\partial x}, \frac{\partial f_3}{\partial y}$$

at the origin. What do you notice?

Now compute the partials $\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}$ at the origin. Even though the picture for f_2 looks qualitatively different than the other two at the origin, can you think of a reason why the behavior of the partials of all three at the origin are the same?

In the problem session, we defined a **singularity** as a point on a curve C_f where both the partials of f are zero. All three curves above have singularities at the origin. Curves (1) and (3) have singularities where the curve crosses itself. Curve (2) has a singularity which is a "cusp", or sharp point.